THE GEOMETRY OF SPACETIME WITH SUPERLUMINAL PHENOMENA

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Abstract. Recent theoretical results show the existence of arbitrary speeds ($0 \leq v < \infty$) solutions of all relativistic wave equations. Some recent experiments confirm the results for sound waves. The question arises naturally: What is the appropriate geometry of spacetime to describe superluminal phenomena? In this paper we present a spacetime model that incorporates the valid results of Relativity Theory and yet describes coherently superluminal phenomena without paradoxes.

1. INTRODUCTION.

Recently it was found that wave equations admit solutions which describe waves propagating slower or faster than the velocity appearing in the equation in question ([1–6]), and there are experiments proving the existence of such waves in the case of sound (supersonic waves) [7]. As a particular case, the Maxwell equations, too, admit subluminal and superluminal wave solutions with arbitrary speed. If such superluminal phenomena exist in Nature then we must reapprise our notions about synchronization, future, past etc. The need of synchronizations different from the standard one emerged from the point of view of tachyons [8–10] but the possibility of superluminal phenomena offers another way.

Now we try to establish the structure of spacetime deriving from the existence of superluminal phenomena. Our treatment is somewhat different from the usual approaches based on coordinates and transformation rules. The mathematical structure of general relativity based on global analysis on manifolds taught us that instead of relative quantities (coordinates, electric and magnetic field etc) and their transformation rules, we have to work with absolute quantities (spacetime, electromagnetic field etc.) and their splitting according to observers (in time and space, in electric and magnetic field etc.). There are such treatments of non-relativistic spacetime and special relativistic spacetime [11–14] which show very well that the point of view of absolute objects admits a clear and simple presentation and excludes the possibility of misunderstanding because the rigorous mathematical structure rules out intuitive notions. In the usual approach observers (reference frames), coordinate systems are intuitive notions and one uses "natural" tacit assumptions. A good example for a misleading tacit assumption is that "if he moves at velocity v relative to you then you moves at velocity $-v$ relative to him". It turns out, however, that this does not hold in the special relativistic spacetime (see ref. [12], § II.4.2); the velocity addition paradox [15] is the consequence of this incorrect tacit assumption.

It is often emphasized that coordinates are labels, not physical entities. On the contrary, splitting of spacetime, spacetime vectors, tensors etc. has a physical meaning: the split quantities describe how an observer perceives absolute objects (the splitting of spacetime by an observer gives the time and the space of the observer, the splitting of electromagnetic field gives the observed electric and magnetic field etc.)

2. PRELIMINARIES.

We intend to define a mathematical model of spacetime based on experimental facts and theoretical assumptions. The basic experimental facts regarding inertial reference frames (observers) are the following. Observers in different inertial reference frames measure time by the "same" clocks and synchronization process and measure space by the "same" rods. The term "same" means a prescription such as: time is measured by the oscillations of a cubic crystal consisting of a given number of molecules of a given material (e.g. quartz), and space is measure by a sideline of that crystal. Then it is found that

1. Time has
   a) a one dimensional affine structure (time translations are meaningful),
   b) an orientation (past and future are distinct);

2. Space has
   a) a three dimensional affine structure (space translations are meaningful),
   b) an orientation (right and left are distinct by the decay of K mesons),
   c) a Euclidean structure (distances and angles are meaningful).

3. The affine structures of time and space are related to each other by uniform motions on straight lines; uniform motion relative to an inertial reference frame seems a uniform motion relative to another inertial reference frame also.

4. Time and space in a given inertial reference frame are related to time and space of other reference frames (transformation rules).

Then 1, 2a, 2b and 3 suggest that spacetime is a four dimensional oriented affine space.

The other structures are deduced from 3 and 4; the different spacetime models come from the different meaning of Euclidean structures on the inertial reference frame spaces and from the transformation rules. However, instead of the explicit use of the transformation rules it is convenient to refer to simpler and more transparent facts expressed in the transformation rules. For instance, in the non-relativistic case we accept

4NR. Absolute time and absolute Euclidean structure (on absolute simultaneous spacetime points) exist.

In the special relativistic case we accept that

4SR. There exists a propagation mode of electromagnetic field configurations, solution of Maxwell equations, and denominated light, which is absolute (independent of the source) and is described by a Lorentzian structure (involving the Euclidean structure).

Then in the non-relativistic spacetime model (NRM) and in the special relativistic spacetime model (SRM) built up on the corresponding assumptions, it becomes a quasi trivial fact that 4NR and 4SR imply the Galilean and the Lorentzian transformation rules, respectively (see [12], § I.8.2.5 and § II.7.1.6.).

Light phenomena are not well described in the NRM; superluminal phenomena are not well described in the SRM. Thus if we want to treat superluminal phenomena, we have to construct a new spacetime model which, similarly to the known cases, will be built up on straightforward theoretical assumptions resulting in a definite transformation rule. Now we accept that

4W. a) Light propagation (in the luminal model—see § 3.2) is absolute (independent of source),
b) There exist electromagnetic field configurations (EFC) that can propagate at arbitrary speeds with respect to material objects (observers),
c) there are electromagnetic field configurations which cannot be at rest with respect to material objects (observers).

4Wb and 4Wc are suggested by the recent theoretical investigations of [2–6] showing the existence of arbitrary speeds ($o \leq v < \infty$) solutions of Maxwell equations.

3. CONSTRUCTION OF A NEW SPACETIME MODEL.

3.1 Absolute simultaneity.

As it is mentioned, we start with the fact that spacetime is a four dimensional oriented affine space M (over the vector space $\mathbf{M}$).

The possibility of EFC of arbitrary speeds allows us to establish an absolute simultaneity S on M by a limit procedure using EFC whose speed tends to infinity. Absolute simultaneity is an equivalence relation on M; then the set of simultaneity classes, $I := M/S$ is **absolute time**, the canonical surjection $\tau : M \to I$ is **time evaluation**.

It is not a hard assumption that simultaneity classes are parallel hyperplanes, which implies the existence of a three dimensional linear subspace $\mathbf{E}$ of $\mathbf{M}$ such that $\tau(x) = \tau(x+\mathbf{E})$ for all $x \in M$. Then $\mathbf{I} := \mathbf{M}/\mathbf{E}$ is a one dimensional vector space and I becomes an affine space over $\mathbf{I}$ by the subtraction $(x+\mathbf{E}) - (y+\mathbf{E}) := x - y + \mathbf{E}$. Then the time evaluation τ will be an affine map over the canonical linear surjection $\boldsymbol{\tau} : \mathbf{M} \to \mathbf{I}$. Keep in mind that $\mathbf{E}$ is the kernel of $\boldsymbol{\tau}$, i.e. $\boldsymbol{\tau} \cdot \boldsymbol{x} = 0$ if and only if $\boldsymbol{x} \in \mathbf{E}$.

$\mathbf{E}$ is the vector space of **absolute spacelike** vectors; from property 2b in the previous paragraph we accept that there is an orientation on $\mathbf{E}$. The orientation of $\mathbf{M}$ and the orientation of $\mathbf{E}$ determine an orientation of $\mathbf{I}$ as follows. Let $(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be a positively oriented basis of $\mathbf{M}$ such that $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is a positively oriented basis of $\mathbf{E}$. Then $\boldsymbol{\tau} \cdot \boldsymbol{e}_0$ is considered to be positive in $\mathbf{I}$. It is not hard to see that the definition of the orientation of $\mathbf{I}$ does not depend on the basis. The orientation of $\mathbf{I}$ gives the orientation (an ordering) of I which we interpret expressing future and past: t' is *later* than t if $t' - t > 0$.

Recapitulating our results, we have

- spacetime, a four dimensional oriented affine space M (over the vector space $\mathbf{M}$),
- absolute time, a one dimensional oriented affine space I (over the vector space $\mathbf{I}$),
- time evaluation, an affine surjection $\tau : M \to I$ (over the linear map $\boldsymbol{\tau} : \mathbf{M} \to \mathbf{I}$),

and $\mathbf{E} := \mathrm{Ker}\boldsymbol{\tau}$.

We call attention to the following fact: in usual treatments time is considered to be the real line but, evidently, e.g. the real number 3 is neither a time point nor a time period; we have got that time is an oriented one dimensional affine space I and time periods are positive elements of the oriented one dimensional vector space $\mathbf{I}$; we shall see that distances, too, will be positive elements of an oriented one dimensional vector space $\mathbf{D}$. Oriented one dimensional vector spaces will be called **measure lines**. We need the products and quotients of elements of different measure lines; e.g. if $m \in \mathbf{D}$ and $s \in \mathbf{I}$, we need m/s. There is a convenient mathematical expression of such products and quotients (see Introduction of ref. [11]) which is not detailed here because formally we can apply the well known rules of multiplication and division.

3.2 Absolute velocities.

We have got M, I and τ which form a part of NRM (see § I.1 of [11]); so we can use all the notions of NRM that do not refer to the Euclidean structure. In particular, $r : I \to M$ is a *world line function*, if $\tau(r(t)) = t$ for all $t \in I$. Then its derivative, the absolute velocity has the property $\boldsymbol{\tau} \cdot \dot{r}(t) = 1$; correspondingly,

$$(1) \qquad V(1) := \left\{ \boldsymbol{u} \in \frac{\mathbf{M}}{\mathbf{I}} \;\middle|\; \boldsymbol{\tau} \cdot \boldsymbol{u} = 1 \right\}$$

is the set of **absolute velocities**.

In contrast to the NRM, in our theory not all world lines are allowed as histories of mass points. According to our assumption 4W.c, the possible particle velocities form a non void subset P of $V(1)$. The elements of P, ∂P and $V(1) \setminus \overline{P}$ are called *particle (or subluminal)* velocities, *luminal* velocities and *superluminal velocities*, respectively. Keep in mind that here velocity means absolute velocity.

We suppose that P is open and connected. A **reference frame** is a smooth map $\boldsymbol{U} : M \to P$.[1] Then the **space of the reference frame** is as in NRM: it consists of the integral curves of the vector field $\boldsymbol{U}$. Each one of the integral lines of $\boldsymbol{U}$ is called an **observer**. **Inertial reference frames** are the ones having constant value. In the following we shall deal only with inertial reference frames, so we omit the term inertial, and we refer to an inertial observer by its constant value, so we say, e.g., an observer $\boldsymbol{u} \in P$. The $\boldsymbol{u}$-space consists of the straight lines parallel to $\boldsymbol{u}$; thus a $\boldsymbol{u}$-space point is of the form $x + \boldsymbol{u} \otimes \mathbf{I}$ for some $x \in M$.

3.3 Observer times.

The flow of time as registered by physical clocks are different depending on the motion of the clocks. (This is an experimental fact [14,17].) Consider the world lines of two (pointlike) clocks with velocity $\boldsymbol{u}$ and $\boldsymbol{u}_o$, respectively. Establish a synchronization of the clocks by an "infinitely" fast superluminal signal. Later the synchronization is repeated, and it is found that the times registered by to the clocks between the two synchronizations are different. Because of the affine structure of observer times (property 1a in § 2) this means that to every $\boldsymbol{u} \in P$ there is a positive number $\kappa_{\boldsymbol{u}}$ in such a way that the time elapsed between the absolute timepoints t_1 and t_2 along the world line with velocity value $\boldsymbol{u}$ equals $\frac{t_2 - t_1}{\kappa_{\boldsymbol{u}}}$.

Now we conceive that the observers in $\boldsymbol{u}$ considers time I to have an affine structure with the $\boldsymbol{u}$-subtraction

$$(2) \qquad (t_2 - t_1)_{\boldsymbol{u}} := \frac{t_2 - t_1}{\kappa_{\boldsymbol{u}}}$$

[1]This definition is the one used in [13,14,15,12]. Note that what we define as a reference frame has been called observer in [11,12].

3.4 Observer spaces.

Spaces of different reference frames are different. However, all the reference frames spaces can be made an affine space over the same vector space $\mathbf{E}$. Take two $\boldsymbol{u}$-lines q_1 and q_2 (representing the endpoints of a rod resting in $\boldsymbol{u}$-space). Then the vector between simultaneous points of q_2 and q_1 is independent of time. In NRM where the Euclidean structure is taken to be absolute, this vector is accepted to be the difference of q_2 and q_1, defining the affine structure of the observer space. Now we take into account that the Euclidean structure depends on observers. Let us consider two sets of observers, in two frames $\boldsymbol{u}_o$ and $\boldsymbol{u}$, both having a resting rod of the same length d (the number of molecules of the given crystal along the rod is the same). Now the observer in $\boldsymbol{u}$ marks the endpoints of the $\boldsymbol{u}_o$-rod at a given instant (i.e. simultaneously) and measures the distance between the marks and finds eventually that it does not equal d. Thus the two observers assign different vectors in $\mathbf{E}$ to the "same" rod.

In view of the fact 2a in paragraph 2, we assume that this difference can be expressed by a linear map which means the following.

To every $\boldsymbol{u} \in P$ there is given a linear bijection $A_{\boldsymbol{u}} : \mathbf{E} \to \mathbf{E}$ in such a way that the observer space $E_{\boldsymbol{u}}$ (the set of straight lines parallel to $\boldsymbol{u}$) is equipped with an affine structure by the subtraction

$$q_2 - q_1 := A_u \cdot (x_2 - x_1) \qquad (x_2 \in q_2, x_1 \in q_1, \tau(x_2) = \tau(x_1)) \tag{3}$$

Since $\mathbf{E}$ is oriented, there is an $\mathbf{E}\wedge\mathbf{E}\wedge\mathbf{E}$ valued canonical translation invariant measure on $\mathbf{E}$ such that the polyhedron spanned by the positively oriented basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ equals $\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3$. $\mathbf{E} \wedge \mathbf{E} \wedge \mathbf{E}$ is an oriented one dimensional vector space, so we can take its cubic root $\mathbf{D}$ (see § IV.4. of [11]). Evidently, the elements of $\mathbf{E} \wedge \mathbf{E} \wedge \mathbf{E}$ are interpreted as volume values, so the elements of $\mathbf{D}$ are distances.

According to item 2c in paragraph 2, every observer $\boldsymbol{u}$ has a Euclidean structure $\mathbf{b}_{\boldsymbol{u}}$. If $\boldsymbol{r}$ and $\boldsymbol{r}_o$ are the same (arbitrary) rods in $\boldsymbol{u}$-space and $\boldsymbol{u}_o$-space, respectively, then they have the same length according to $\boldsymbol{u}$ and $\boldsymbol{u}_o$, respectively. Thus the Euclidean structures of the observer spaces define a unique Euclidean structure $\mathbf{b} : \mathbf{E} \times \mathbf{E} \to \mathbf{D} \otimes \mathbf{D}$ such that $\mathbf{b}_{\boldsymbol{u}}(\boldsymbol{r}, \boldsymbol{r}) = \mathbf{b}(A_{\boldsymbol{u}} \cdot \boldsymbol{r}, A_{\boldsymbol{u}} \cdot \boldsymbol{r})$.

3.5 Continuity.

The specific meaning of the set of particle velocities P in $V(1)$ is reflected by the fact, that we require $\boldsymbol{u} \to \kappa_{\boldsymbol{u}}$ and $\boldsymbol{u} \to A_{\boldsymbol{u}}$ to be continuous and continuously inextensible to the points of ∂P in such a way that they remain positive and non-degenerate, respectively.

3.6 The new spacetime model.

Recapitulating our results, we see that we have got the Euclidean structure on $\mathbf{E}$, so all the items of NRM are present, and further structures are introduced. We have as a new spacetime model

$$(M, I, \tau, \mathbf{D}, \mathbf{b}, P, \kappa, A)$$

where

- $(M, I, \tau, \mathbf{D}, \mathbf{b})$ is a NRM (in which the set $V(1)$ of absolute velocities is defined), and
- $P \subset V(1)$ is a nonvoid connected open subset,
- $\kappa : P \to \mathbb{R}^+, \quad \boldsymbol{u} \mapsto \kappa_{\boldsymbol{u}}$,
- $A : P \to \mathrm{GL}(\mathbf{E}), \quad \boldsymbol{u} \mapsto A_{\boldsymbol{u}}$

are continuous functions which cannot be extended continuously to the points of ∂P.

3.7 Notations.

The action of a linear map is denoted by a dot e.g $\boldsymbol{\tau} \cdot \boldsymbol{x}$. In the following, instead of $\mathbf{b}$ we shall write a dot product, too, i.e. $\boldsymbol{q} \cdot \boldsymbol{r} := \boldsymbol{b}(\boldsymbol{q}, \boldsymbol{r})$; furthermore, we put $|\boldsymbol{q}|^2 := \boldsymbol{b}(\boldsymbol{q}, \boldsymbol{q})$ for $\boldsymbol{qr} \in \mathbf{E}$, and similar notations will be applied for the induced Euclidean structure on $\frac{\mathbf{E}}{\mathbf{I}}$ etc (see § I.1.4.2 of [11]). If one treats linear maps and bilinear maps as tensors then all these dots correspond to contractions and no ambiguity arises.

4. SOME FORMULAE IN THE NEW SPACETIME MODEL.

4.1. Splitting of spacetime.

Observers in the reference frame $\boldsymbol{u} \in P$ splits spacetime into time and space in such a way that to a spacetime point x they assigns the corresponding absolute timepoint and the $\boldsymbol{u}$-spacepoint that x is incident with, i.e. the **$\boldsymbol{u}$-splitting of spacetime** is the map

$$H_{\boldsymbol{u}} : M \to I \times E_{\boldsymbol{u}}, \qquad x \mapsto (x + \mathbf{E}, x + \boldsymbol{u} \otimes \mathbf{I}). \tag{4}$$

This splitting is the same as in NRM. However, since the affine structure of $E_{\boldsymbol{u}}$ differs from that in NRM, and we have to consider the affine structure of I depending on the observer (see 3.4.),now we find that $H_{\boldsymbol{u}}$ is an affine map over the linear map

$$\boldsymbol{s}_{\boldsymbol{u}} : \mathbf{M} \to \mathbf{I} \times \mathbf{E}, \qquad \boldsymbol{x} \mapsto \left(\frac{\boldsymbol{\tau} \cdot \boldsymbol{x}}{\kappa_{\boldsymbol{u}}}, A_{\boldsymbol{u}} \cdot \big(\cdot \boldsymbol{x} - (\boldsymbol{\tau} \cdot \boldsymbol{x}) \boldsymbol{u} \big) \right) \tag{5}$$

which we call the **$\boldsymbol{u}$-splitting of vectors**.

4.2 Relative velocities.

A world line function represents the history of a mass point or a light ray signal in spacetime. An observer perceives this history as a motion. The motion relative to the reference frame $\boldsymbol{u} \in P$ corresponding to the world line function r is described by the function $r_{\boldsymbol{u}}$ which assigns to a timepoint t the $\boldsymbol{u}$-space point that $r(t)$ is incident with:

$$r_{\boldsymbol{u}} : I \to E_u, \qquad t \mapsto r(t) + \boldsymbol{u} \otimes \mathbf{I}. \tag{6}$$

The velocity of the motion relative to the observer is obtained by

$$\lim_{t_2 \to t_1} \frac{r_{\boldsymbol{u}}(t_2) - r_{\boldsymbol{u}}(t_1)}{(t_2 - t_1)_{\boldsymbol{u}}} = \kappa_{\boldsymbol{u}} A_{\boldsymbol{u}} \cdot (\dot{r}(t_1) - \boldsymbol{u}). \tag{7}$$

That is why we accept that if $\boldsymbol{w} \in V(1)$ and $\boldsymbol{u} \in P$ then

$$\boldsymbol{v}_{\boldsymbol{wu}} := \kappa_{\boldsymbol{u}} A_u \cdot (\boldsymbol{w} - \boldsymbol{u}) \tag{8}$$

is the **relative velocity of $\boldsymbol{w}$ with respect to $\boldsymbol{u}$.**

Then we have for $\boldsymbol{u}, \boldsymbol{u}' \in P$

$$\kappa'_{\boldsymbol{u}} A_{\boldsymbol{u}}^{-1} \cdot \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} = -\kappa_{\boldsymbol{u}} A_{\boldsymbol{u}'}^{-1} \cdot \boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'} \tag{9}$$

which implies, in general, that $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'} \neq -\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$.

Furthermore, we easily find the **velocity addition formula**: if $\boldsymbol{w} \in V(1)$ and $\boldsymbol{u}, \boldsymbol{u}' \in P$ then

$$\boldsymbol{v}_{\boldsymbol{wu}} = \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} + \frac{\kappa_{\boldsymbol{u}}}{\kappa_{\boldsymbol{u}'}} A_{\boldsymbol{u}} \cdot A_{\boldsymbol{u}'}^{-1} \cdot \boldsymbol{v}_{\boldsymbol{w}\boldsymbol{u}'} \tag{10}$$

or

$$\frac{\kappa_{\boldsymbol{u}'}}{\kappa_{\boldsymbol{u}}} A_{\boldsymbol{u}'} \cdot A_{\boldsymbol{u}}^{-1} \cdot \boldsymbol{v}_{\boldsymbol{wu}} = \boldsymbol{v}_{\boldsymbol{w}\boldsymbol{u}'} - \boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}. \tag{11}$$

4.3 Comparison of splittings.

Let us compare now the splittings in two different reference frames $\boldsymbol{u}, \boldsymbol{u}' \in P$ which is expressed by $\boldsymbol{s}_{\boldsymbol{u}'} \cdot \boldsymbol{s}_{\boldsymbol{u}}^{-1}$. Since $\boldsymbol{s}_{\boldsymbol{u}}^{-1}(\boldsymbol{t}, \boldsymbol{q}) = A_{\boldsymbol{u}}^{-1} \cdot \boldsymbol{q} + \frac{\boldsymbol{t}}{\kappa_{\boldsymbol{u}}} \boldsymbol{u}$, we easily find the **vector transformation law**:

$$\boldsymbol{s}_{\boldsymbol{u}'} \cdot \boldsymbol{s}_{u}^{-1}(\boldsymbol{t}, \boldsymbol{q}) = \left(\frac{\kappa_{\boldsymbol{u}}}{\kappa_{\boldsymbol{u}'}} \boldsymbol{t}, A_{\boldsymbol{u}'} \cdot A_{\boldsymbol{u}}^{-1} \cdot \boldsymbol{q} + \boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'} \frac{\kappa_{\boldsymbol{u}}}{\kappa'_{\boldsymbol{u}}} \boldsymbol{t} \right); \qquad (\boldsymbol{t}, \boldsymbol{q}) \in \mathbf{I} \times \mathbf{E}. \tag{12}$$

We call the reader's attention to the fact that here $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}$ cannot be substituted with $-\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}$; if we want to use the latter quantity, we obtain $A_{\boldsymbol{u}'} \cdot A_{\boldsymbol{u}}^{-1} \cdot (\boldsymbol{q} - \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \boldsymbol{t})$ in the formula of the transformation law.

5. THE LORENTZ AETHER MODEL (LAM) [18,19].

5.1 Aether, dilation, contraction.

The previous type of spacetime model is very general (it contains the NRM as a particular case: then $P = V(1)$, $A_{\boldsymbol{u}}$ is the identity of $\mathbf{E}$ and $\kappa_{\boldsymbol{u}} = 1$ for all $\boldsymbol{u}$). Now we shall detail a special model (LAM) where

- there are an $\boldsymbol{u}_o \in P$ and a $0 < c \in \frac{\mathbf{D}}{\mathbf{I}}$ such that

$$P = \left\{ \boldsymbol{u} \in \frac{\mathbf{M}}{\mathbf{I}} \;\middle|\; |\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}| < c \right\} = \boldsymbol{u}_o + \left\{ \boldsymbol{v} \in \frac{\mathbf{E}}{\mathbf{I}} \;\middle|\; |\boldsymbol{v}| < c \right\}, \tag{13}$$

$$\kappa_{\boldsymbol{u}} = \frac{1}{\sqrt{1 - \frac{|\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}|^2}{c^2}}} \tag{14}$$

$$A_{\boldsymbol{u}} = \mathbf{1} - (1 - \kappa_{\boldsymbol{u}}) \frac{\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}}{|\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}|} \otimes \frac{\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}}{|\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}|} \tag{15}$$

Regarding the previous definition, note that the symbol $\mathbf{1}$ denotes the identity map of $\mathbf{E}$ and for $\boldsymbol{n}$ the linear map $\boldsymbol{n} \otimes \boldsymbol{n}$ acts as $\boldsymbol{q} \mapsto \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{q})$.

We find that $\kappa_{\boldsymbol{u}_o} = 1$; furthermore if $\boldsymbol{u} = \boldsymbol{u}_o$ then the expression containing $|\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}| = 0$ in the denominator is meaningless but it is multiplied by zero, so we mean that $A_{\boldsymbol{u}_o} = \mathbf{1}$.

Of course, the set of luminal velocities is

$$\partial P = \left\{ \boldsymbol{w} \in \frac{\mathbf{M}}{\mathbf{I}} \;\middle|\; |\boldsymbol{v}_{\boldsymbol{w}\boldsymbol{u}_o}| = c \right\}. \tag{16}$$

The reference frame with constant velocity $\boldsymbol{u}_o$ is called the **aether**, c is the **light speed in the aether**, $\kappa_{\boldsymbol{u}}$ is the **time dilation factor** corresponding to $\boldsymbol{u}$, and

$$A_{\boldsymbol{u}}^{-1} = \mathbf{1} + \frac{1 - \kappa_{\boldsymbol{u}}}{\kappa_{\boldsymbol{u}}} \frac{\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}}{|\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}|} \otimes \frac{\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}}{|\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}|} \tag{17}$$

is the **Lorentz contraction map** corresponding to $\boldsymbol{u}$: $|A_{\boldsymbol{u}}^{-1} \cdot \boldsymbol{q}| = |\boldsymbol{q}|$ if $\boldsymbol{q}$ is orthogonal to $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}$ and $|A_{\boldsymbol{u}}^{-1} \cdot \boldsymbol{q}| = \kappa_{\boldsymbol{u}}|\boldsymbol{q}|$ if $\boldsymbol{q}$ is parallel to $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}$.

5.2 Relative velocities.

The equality

$$\boldsymbol{v}_{\boldsymbol{w}\boldsymbol{u}_o} = \boldsymbol{w} - \boldsymbol{u}_o \tag{18}$$

is a trivial fact for $\boldsymbol{w} \in V(1)$; in general, if $\boldsymbol{u} \in P$ then

$$\boldsymbol{v}_{\boldsymbol{w}\boldsymbol{u}} = \kappa_{\boldsymbol{u}} A_u \cdot (\boldsymbol{v}_{\boldsymbol{w}\boldsymbol{u}_o} - \boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}). \tag{19}$$

Having the LAM, we can calculate quite easily all the quantities appearing in usual applications of aether theory [18–26] without further assumptions and heuristic considerations. For instance, we have for $\boldsymbol{w} \in V(1)$, $\boldsymbol{u} \in P$

$$|\boldsymbol{v}_{\boldsymbol{w}\boldsymbol{u}}|^2 = \kappa_{\boldsymbol{u}}^2 \left[|\boldsymbol{v}_{\boldsymbol{w}\boldsymbol{u}_o}|^2 + \kappa_{\boldsymbol{u}}^2 \left(|\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}|^2 - 2\boldsymbol{v}_{\boldsymbol{w}\boldsymbol{u}_o} \cdot \boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o} + \frac{(\boldsymbol{v}_{\boldsymbol{w}\boldsymbol{u}_o} \cdot \boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o})^2}{c^2} \right) \right]. \tag{20}$$

We see that for $\boldsymbol{u}, \boldsymbol{u}' \in P$

$$|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}| \neq |\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}| \qquad \text{in general,} \tag{21}$$

more closely,

$$|\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}| = |\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}'}| \qquad \text{if and only if} \qquad |\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}_o}| = |\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}|. \tag{22}$$

In particular, we have

$$|\boldsymbol{v}_{\boldsymbol{u}_o\boldsymbol{u}}| = \kappa_u^2 |\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}|. \tag{23}$$

5.3 Ives-Tangherlini-Marinov coordinates [16,18,19,21,22,23].

If we choose a positively oriented basis $(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ in $\mathbf{M}$ such that $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is a positively oriented orthogonal basis in $\mathbf{E}$, $\boldsymbol{e}_0$ is parallel to $\boldsymbol{u}_o$, $\boldsymbol{e}_1$ is parallel to $-\boldsymbol{v}_{\boldsymbol{u}_o\boldsymbol{u}}$ (which is not equal to $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}$!) then the transformation law given in 4.1 applied to $\boldsymbol{u}' := \boldsymbol{u}_o$ and expressed in coordinates relative to the chosen basis coincides with the well known Ives-Tangherlini-Marinov transformation.

6. THE RELATIVISTIC STRUCTURE DUE TO THE AETHER.

6.1 The Lorentz form.

Due to the privileged observer (aether) in the LAM we can introduce a Lorentz form on $\mathbf{M}$ by the use of $\boldsymbol{u}_o$-splitting:

$$\boldsymbol{x} \cdot \boldsymbol{y} := (\boldsymbol{x} - (\boldsymbol{\tau} \cdot \boldsymbol{x})\boldsymbol{u}_o) \cdot (\boldsymbol{y} - (\boldsymbol{\tau} \cdot \boldsymbol{y})\boldsymbol{u}_o) - c^2(\boldsymbol{\tau} \cdot \boldsymbol{x})(\boldsymbol{\tau} \cdot \boldsymbol{y}). \tag{24}$$

The Lorentz product denoted by a dot on the left hand side is an extension of the Euclidean dot product appearing on the right hand side, so the notation is consistent.

The Lorentz form is arrow oriented in such a way that $\boldsymbol{u}_o$ is pointing to the future.

So $(M, \mathbf{D}, \cdot)$ is a SRM *associated* to the LAM in which all the well known relativistic notions can be used ([11], Part II).

6.2 Relativistic splitting.

For $\boldsymbol{w}, \boldsymbol{w}' \in V(1)$, we have

$$-\boldsymbol{w}' \cdot \boldsymbol{w} = c^2 - \boldsymbol{v}_{\boldsymbol{w}'\boldsymbol{u}_o} \cdot \boldsymbol{v}_{\boldsymbol{w}\boldsymbol{u}_o}. \tag{25}$$

In particular, $-\boldsymbol{u}_o \cdot \boldsymbol{w} = c^2$ for all $\boldsymbol{w} \in V(1)$. Moreover, it follows that

$$\left\{ \hat{\boldsymbol{u}} \in \frac{\mathbf{M}}{\mathbf{D}} \;\middle|\; \hat{\boldsymbol{u}} \text{ is future directed }, -\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{u}} < 1 \right\} = \left\{ \frac{\kappa_{\boldsymbol{u}}\boldsymbol{u}}{c} \;\middle|\; \boldsymbol{u} \in P \right\}. \tag{26}$$

As a consequence, the inertial reference frames of the LAM coincide with the inertial reference frames of the associated SRM. For the sake of brevity, we introduce the notation

$$\hat{\boldsymbol{u}} := \frac{\kappa_u \boldsymbol{u}}{c} \qquad (\boldsymbol{u} \in P). \tag{27}$$

We mention that $\hat{\boldsymbol{u}}_o = -\frac{\boldsymbol{u}_o}{c}$ and

$$\boldsymbol{\tau} \cdot \boldsymbol{x} = -\frac{\boldsymbol{u}_o \cdot \boldsymbol{x}}{c^2} \qquad (\boldsymbol{x} \in \mathbf{M}). \tag{28}$$

The relativistic synchronization established by luminal phenomena depends on the reference frame. The reference frame $\boldsymbol{u} \in P$ finds that luminally simultaneous spacetime points are hyperplanes parallel to the three dimensional subspace

$$\mathbf{E}_{\boldsymbol{u}} := \{\boldsymbol{x} \in \mid \boldsymbol{u} \cdot \boldsymbol{x} = 0\}. \tag{29}$$

Using this synchronization, the space of the reference frame $\boldsymbol{u}$ (the set of straight lines parallel to $\boldsymbol{u}$) becomes an affine space over $\mathbf{E}_{\boldsymbol{u}}$ by the subtraction

$$(q_2 - q_1)_{rel} := x_2 - x_1 \qquad (x_2 \in q_2, x_1 \in q_1, \boldsymbol{u} \cdot (x_2 - x_1) = 0). \tag{30}$$

Thus $\boldsymbol{u}$-space vectors are different in the LAM and in the associated SRM. This important fact disappears when considering coordinates, since coordinates of arbitrary three dimensional vector spaces are triplets of numbers.

The set $I_{\boldsymbol{u}}$ of hyperplanes parallel to $\mathbf{E}_{\boldsymbol{u}}$ constitute the time of the reference frame; this is a one dimensional affine space over $\mathbf{I}$ by the subtraction

$$t_2 - t_1 := -\frac{\hat{\boldsymbol{u}}}{c} \cdot (x_2 - x_1) \qquad (x_2 \in t_2, x_1 \in t_1). \tag{31}$$

According to the relativistic synchronization, the observers in the reference frame $\boldsymbol{u}$ split spacetime in time and space by

$$H_{\hat{\boldsymbol{u}}} : M \to I_{\boldsymbol{u}} \times E_{\boldsymbol{u}}, \qquad x \mapsto (x + E_{\boldsymbol{u}}, x + \boldsymbol{u} \otimes I) \tag{32}$$

which is an affine mapping over the linear map

$$\boldsymbol{h}_{\boldsymbol{u}} : \mathbf{M} \to \mathbf{I} \times \mathbf{E}_{\boldsymbol{u}}, \qquad \boldsymbol{x} \mapsto \left(\frac{-\hat{u} \cdot \boldsymbol{x}}{c}, \boldsymbol{x} + (\hat{u} \cdot \boldsymbol{x})\hat{u} \right). \tag{33}$$

As an important fact, we mention that the relativistic relative velocity of $\boldsymbol{u}' \in P$ with respect to $\boldsymbol{u} \in P$ is (see [11], § II.4.2.)

$$\boldsymbol{v}_{\hat{\boldsymbol{u}}'\hat{\boldsymbol{u}}} := c \left(\frac{\hat{\boldsymbol{u}}'}{-\hat{\boldsymbol{u}} \cdot \boldsymbol{u}'} - \hat{\boldsymbol{u}} \right). \tag{34}$$

6.3 Lorentz boosts.

The relativistic spaces of different reference frames $\boldsymbol{u}$ and $\boldsymbol{u}'$ are affine spaces over the different vector spaces $\mathbf{E}_{\boldsymbol{u}}$ and $\mathbf{E}_{\boldsymbol{u}'}$, respectively. However, there is a "canonical" linear bijection between $\mathbf{E}_{\boldsymbol{u}}$ and $\mathbf{E}_{\boldsymbol{u}'}$ which can be used to identify these different vector spaces; this linear bijection is the **Lorentz boost** from $\boldsymbol{u}$ to $\boldsymbol{u}'$ (see § II.1.3.8 of [12])

$$L(\boldsymbol{u}', \boldsymbol{u}) := \mathbf{1} + \frac{(\hat{\boldsymbol{u}}' + \hat{\boldsymbol{u}}) \otimes (\hat{\boldsymbol{u}}' + \hat{\boldsymbol{u}})}{1 - \hat{\boldsymbol{u}}' \cdot \hat{\boldsymbol{u}}} - 2\hat{\boldsymbol{u}}' \otimes \hat{\boldsymbol{u}} \tag{35}$$

where $\mathbf{1}$ is the identity map of $\mathbf{M}$.

This linear bijection preserves the Lorentz form and its arrow orientation as well as the orientation of spacetime. Moreover, we have

$$
\begin{aligned}
&L(\boldsymbol{u}',\boldsymbol{u})\cdot\hat{\boldsymbol{u}}=\hat{\boldsymbol{u}}'\\
&L(\boldsymbol{u}',\boldsymbol{u})\cdot\boldsymbol{q}=\boldsymbol{q}\quad\text{if }\boldsymbol{q}\in\mathbf{E}_{\boldsymbol{u}}\cap\mathbf{E}_{\boldsymbol{u}'}\\
&L(\boldsymbol{u}',\boldsymbol{u})\cdot\boldsymbol{v}_{\hat{\boldsymbol{u}}'\hat{\boldsymbol{u}}}=-\boldsymbol{v}_{\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}'}\\
&L(\boldsymbol{u}',\boldsymbol{u})^{-1}=L(\boldsymbol{u},\boldsymbol{u}')
\end{aligned}
\tag{36}
$$

In usual treatments based on coordinates the space of every reference frame is considered to consist of the elements of form $(0,\xi_1,\xi^2,\xi^3)$. This corresponds to the fact that one chooses a reference frame ("rest frame") and implicitly all the other reference frames spaces are Lorentz boosted to the space of that reference frame.

6.4 Comparison of splittings.

If superluminal phenomena do exist then the Lorentz aether model offers an adequate structure for spacetime. Then we conceive that the relativistic formulae used in physics refer to the SRM associated to the LAM. Therefore it is important to compare the splitting $\boldsymbol{s}_{\boldsymbol{u}}$ in LAM and the splitting $\boldsymbol{h}_{\boldsymbol{u}}$ in SRM due to a reference frame $\boldsymbol{u}\in P$. Since $\boldsymbol{h}_{\boldsymbol{u}}^{-1}(\boldsymbol{t},\boldsymbol{q})=\boldsymbol{q}+\hat{\boldsymbol{u}}c\boldsymbol{t}$ for $\boldsymbol{t}\in\mathbf{I}$, $\boldsymbol{q}\in\mathbf{E}_{\boldsymbol{u}}$, we easily find the comparison:

$$
\boldsymbol{s}_{\boldsymbol{u}}\cdot\boldsymbol{h}_{\boldsymbol{u}}^{-1}(\boldsymbol{t},\boldsymbol{q})=\left(\boldsymbol{t}+\frac{\boldsymbol{\tau}\cdot\boldsymbol{q}}{\kappa_{\boldsymbol{u}}},\boldsymbol{q}-(\boldsymbol{\tau}\cdot\boldsymbol{q})\boldsymbol{u}\right).
\tag{37}
$$

However, this is rarely useful, because relates elements in $\mathbf{E}_{\boldsymbol{u}}$ to elements in $\mathbf{E}=\mathbf{E}_{\boldsymbol{u}_o}$. To have a nicer formula, we map $\mathbf{E}_{\boldsymbol{u}}$ onto $\mathbf{E}_{\boldsymbol{u}_o}$ "canonically", i.e. we apply a Lorentz boost from $\boldsymbol{u}$ to $\boldsymbol{u}_o$, and instead of the splitting $\boldsymbol{h}_{\boldsymbol{u}}$ we consider

$$
\boldsymbol{h}_{\boldsymbol{u}_o\boldsymbol{u}}:=\left(\mathrm{id}_{\mathbf{I}}\times L(\boldsymbol{u}_o,\boldsymbol{u})|_{\mathbf{E}_{\boldsymbol{u}}}\right)\cdot\boldsymbol{h}_{\boldsymbol{u}}=\boldsymbol{h}_{\boldsymbol{u}_o}\cdot L(\boldsymbol{u}_o,\boldsymbol{u})
\tag{38}
$$

(see § II.7.1.4–7.1.6 of [11]) for which

$$
\boldsymbol{h}_{\boldsymbol{u}_o\boldsymbol{u}}\cdot\boldsymbol{x}=\left(-\frac{\hat{\boldsymbol{u}}}{c}\cdot\boldsymbol{x},L(\boldsymbol{u}_o,\boldsymbol{u})\cdot\boldsymbol{x}+(\hat{\boldsymbol{u}}\cdot\boldsymbol{x})\hat{\boldsymbol{u}}\right)
\tag{39}
$$

holds, and we look for the explicit expression of

$$
\boldsymbol{s}_{\boldsymbol{u}}\cdot\boldsymbol{h}_{\boldsymbol{u}_o\boldsymbol{u}}^{-1}=\boldsymbol{s}_{\boldsymbol{u}}\cdot L(\boldsymbol{u},\boldsymbol{u}_o)\cdot\boldsymbol{h}_{\boldsymbol{u}_o}^{-1}
\tag{40}
$$

applied to elements $(\boldsymbol{t},\boldsymbol{q})\in\mathbf{I}\times\mathbf{E}_{\boldsymbol{u}_o}$.

The time component, by $L(\boldsymbol{u}.\boldsymbol{u}_o)\cdot\boldsymbol{h}_{\boldsymbol{u}_o}^{-1}(\boldsymbol{t},\boldsymbol{q}) = L(\boldsymbol{u},\boldsymbol{u}_o)\cdot(\boldsymbol{q}+\boldsymbol{u}_o\boldsymbol{t}) = L(\boldsymbol{u},\boldsymbol{u}_o)\cdot\boldsymbol{q}+\kappa_u\boldsymbol{u}\boldsymbol{t}$, by Eqs. 5, 27 and 34 and by the fact that $\boldsymbol{u}\cdot L(\boldsymbol{u},\boldsymbol{u}_o)\cdot\boldsymbol{q}_o = 0$, is obtained as

$$\boldsymbol{t} + \frac{\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}}{c^2}\cdot\boldsymbol{q}. \tag{41}$$

Furthermore we obtain by simple calculations that

$$A_{\boldsymbol{u}} = L(\boldsymbol{u}_o,\boldsymbol{u})(\mathbf{1}+\hat{\boldsymbol{u}}\otimes\hat{\boldsymbol{u}})|_{\mathbf{E}_{\boldsymbol{u}_o}} \tag{42}$$

and taking into account the formulae

$$\boldsymbol{x} - (\boldsymbol{\tau}\cdot\boldsymbol{x})\boldsymbol{u} = \left(\mathbf{1} + \frac{\hat{\boldsymbol{u}}\otimes\hat{\boldsymbol{u}}_o}{c\kappa_{\boldsymbol{u}}}\right)\cdot\boldsymbol{x}, \tag{43}$$

$$(\mathbf{1}+\hat{\boldsymbol{u}}\otimes\hat{\boldsymbol{u}})\cdot\left(\mathbf{1} + \frac{\hat{\boldsymbol{u}}\otimes\hat{\boldsymbol{u}}_o}{c\kappa_{\boldsymbol{u}}}\right) = \mathbf{1}+\hat{\boldsymbol{u}}\otimes\hat{\boldsymbol{u}}, \tag{44}$$

we find that the space component equals $\boldsymbol{q}$; summarizing our results:

$$\boldsymbol{s}_{\boldsymbol{u}}\cdot\boldsymbol{h}_{\boldsymbol{u}_o\boldsymbol{u}}^{-1}(\boldsymbol{t},\boldsymbol{q}) = \left(\boldsymbol{t} + \frac{\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}}{c^2}\cdot\boldsymbol{q},\ \boldsymbol{q}\right) \qquad \left((\boldsymbol{t},\boldsymbol{q})\in\mathbf{I}\times\mathbf{E}_{\boldsymbol{u}_o}\right). \tag{45}$$

6.5 Comparison of motions in LAM and in SRM.

The history of a masspoint given by a world line function $r : I \mapsto M$ is perceived by an observer in $\boldsymbol{u}$ as a motion; the motion is described in different ways in LAM and in SRM. To get a better comparison between the different descriptions, we consider a vectorization of spacetime by an origin o, i.e. the vectorized motion $\boldsymbol{r}_{\boldsymbol{u}}$ in LAM is obtained from

$$\boldsymbol{s}_{\boldsymbol{u}}(r(t)-o) = (t-t_o, r(t)-o+\boldsymbol{u}(\boldsymbol{t}-t_o)) \tag{46}$$

where $t_o := \tau(o)$; thus by $\boldsymbol{t} := t - t_o$ we get

$$\boldsymbol{r}_{\boldsymbol{u}} : \mathbf{I}\to\mathbf{E}, \qquad \boldsymbol{t}\mapsto r(t_o+\boldsymbol{t}) - o + \boldsymbol{u}\boldsymbol{t} \tag{47}$$

The vectorized motion $\boldsymbol{r}_{\hat{\boldsymbol{u}}}$ in SRM (applying a boost to $\boldsymbol{u}_o$) is obtained from

$$h_{\boldsymbol{u}_o\boldsymbol{u}}(r(t)-o) = \left(-\frac{\hat{\boldsymbol{u}}}{c}\cdot(r(t)-o),\ L(\boldsymbol{u}_o,\boldsymbol{u})\cdot(r(t)-o) + (\hat{\boldsymbol{u}}\cdot(r(t)-o))\hat{\boldsymbol{u}}\right). \tag{48}$$

Since

$$\boldsymbol{t}_{\boldsymbol{u}} := -\frac{\hat{\boldsymbol{u}}}{c}\cdot(r(t)-o) \tag{49}$$

gives the relativistic $\boldsymbol{u}$-time as a function of absolute time t from which we can express t as a function of $\boldsymbol{t_u}$, we get the vectorized motion in SRM:

$$(50) \qquad \boldsymbol{r_{\hat{u}}} : \mathbf{I} \to \mathbf{E}, \qquad \boldsymbol{t_u} \mapsto L(\boldsymbol{u}_o, \boldsymbol{u}) \cdot (r(t(\boldsymbol{t}_u)) - o) + (\hat{\boldsymbol{u}} \cdot (r(t(t_{\boldsymbol{u}}) - o))\hat{\boldsymbol{u}}.$$

Then these formulae or the one at the end of the previous paragraph allow us to recover the LAM motion from the SRM motion: the function $\boldsymbol{t} \mapsto \boldsymbol{t} + \frac{\boldsymbol{v_{uu_o}}}{c^2} \cdot \boldsymbol{r_{\hat{u}}t})$ is continuously differentiable, its derivative is everywhere positive, so it has a continuously differentiable inverse, denoted by $\boldsymbol{s} \mapsto \boldsymbol{t}(\boldsymbol{s})$, and we have

$$(51) \qquad \boldsymbol{r_u}(\boldsymbol{s}) = \boldsymbol{r_{\hat{u}}}(\boldsymbol{t}(\boldsymbol{s}))$$

which implies

$$(52) \qquad \dot{\boldsymbol{r}}_{\boldsymbol{u}}(\boldsymbol{s}) = \frac{\dot{\boldsymbol{r}}_{\hat{\boldsymbol{u}}}(\boldsymbol{t}(\boldsymbol{s}))}{1 + (\boldsymbol{v_{uu_o}}/c^2) \cdot \dot{\boldsymbol{r}}_{\hat{\boldsymbol{u}}}(\boldsymbol{t}(\boldsymbol{s}))}.$$

6.6 Propagation of superluminal waves.

The Lorentz invariance of the Maxwell equations means in our language that the relativistic split form of the absolute Maxwell equations is the same for all observers. Thus time, space and velocity in a solution of the split Maxwell equations concern the relativistic splitting due to an observer $\boldsymbol{u}$. Now we want to express the solution in quantities corresponding to the aether splitting. Since the usual form of the solutions is given in coordinates which means that all the quantities are automatically boosted to a "basic" observer, $\boldsymbol{u}_o$ in our notations, the result of the previous paragraph says us that passing from the relativistic splitting (coordinates) to the aether splitting, space vectors remain unchanged and the relativistic time $\boldsymbol{t}$ is to be substituted with $\boldsymbol{t} - \frac{\boldsymbol{v_{uu_o}}}{c^2} \cdot \boldsymbol{q}$.

Suppose now that we are given a solution of the Maxwell equations relative to the observer $\boldsymbol{u}$, and the solution describes a wave propagating with velocity $\boldsymbol{v}$. The wave propagation The wave propagation corresponds a uniform motion with velocity $\boldsymbol{v}$ in SRM, thus we infer from the result at the end of the previous paragraph that the relative velocity in LAM equals

$$(53) \qquad \frac{\boldsymbol{v}}{1 + \frac{\boldsymbol{v_{uu_o}} \cdot \boldsymbol{v}}{c^2}}.$$

The denominator must be positive, which means that for an observer $\boldsymbol{u} \neq \boldsymbol{u}_o$ not all elements of $\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$ are allowed as relativistic relative velocities. Regarding in the reversed order, we can say that all elements of $\frac{\mathbf{E}_{u_o}}{\mathbf{I}}$ can be relative velocities with respect to an arbitrary observer $\boldsymbol{u}$ in the LAM but their transforms in the associated SRM do not fill the whole $\frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$.

6.7 An application to rotating bodies.

There is a long dispute on whether the Lorentz aether theory or special relativity is the adequate theory of spacetime. If superluminal phenomena will be detected then there is no doubt. If not, the present mathematical model may help us to answer the question ruling out loosely defined notions and tacit assumptions regarding Lorentz aether theory which can be found in most of the reasonings (e.g., in [24,25]) as it is pointed out in [19].

The experiments proposed in [24,25] refer to uniformly rotating rigid bodies. However, as it turns out (see § II.6.7–6.8 of [12]), the relativistic theory does not admit an object which would have all the well known usual properties of a nonrelativistic uniformly rotating rigid body, so we must be very cautious in reasonings regarding them.

Let o be a spacetime point, B_o be a subset of $\mathbf{E} = \mathbf{E}_{\boldsymbol{u}_o}$ and $\Omega_o : \mathbf{E} \to \mathbf{E}$ an antisymmetric linear map. Then the collection of world lines

$$\{\boldsymbol{t} \mapsto o + \kappa_{\boldsymbol{u}} \boldsymbol{u}\boldsymbol{t} + A_{\boldsymbol{u}} \cdot \exp(\boldsymbol{t}\Omega_o) \cdot \boldsymbol{q}_o \mid \boldsymbol{q}_o \in B_o\} \tag{54}$$

gives a uniformly rotating rigid body in the space of the reference frame $\boldsymbol{u}$ according to the LAM.

It seems, the "uniformly rotating reference frame II" described in [12], § II.6.8. is the best candidate to be accepted as a uniformly rotating relativistic rigid body. This describes an object which is seen uniformly rotating relative to $\boldsymbol{u} \in P$. All its points are given by a world line of the form

$$\boldsymbol{t} \mapsto r(\boldsymbol{t}) := o + \hat{\boldsymbol{u}}\boldsymbol{t} + \exp(\boldsymbol{t}\Omega) \cdot \boldsymbol{q} \tag{55}$$

where o is a given spacetime point, Ω is an antisymmetric linear map $\mathbf{E}_{\boldsymbol{u}} \to \frac{\mathbf{E}_{\boldsymbol{u}}}{\mathbf{I}}$ and $\boldsymbol{q} \in \mathbf{E}_{\boldsymbol{u}}$ is in the kernel of Ω, $\boldsymbol{t}$ is the (relativistic) time of the reference frame $\boldsymbol{u}$ passed from the $\boldsymbol{u}$-timepoint corresponding to o; lastly, the inequality $\omega|\boldsymbol{q}| < c$ must be satisfied where $\omega := |\Omega|$.

The $\boldsymbol{u}$-splittings of $\boldsymbol{t} \mapsto r(\boldsymbol{t}) - o$ in SRM and in LAM give the corresponding motion relative to $\boldsymbol{u}$.

The relativistic motion (see § 6.5) is indeed a uniform rotation

$$\boldsymbol{t} \mapsto \exp(\boldsymbol{t}\Omega_o) \cdot \boldsymbol{q}_o \tag{56}$$

where $\Omega_o := L(\boldsymbol{u}, \boldsymbol{u}_o) \cdot \Omega \cdot L(\boldsymbol{u}, \boldsymbol{u}_o)$ and $\boldsymbol{q}_o := L(\boldsymbol{u}, \boldsymbol{u}_o) \cdot \boldsymbol{q}$.

As concerns the motion in LAM, we have to find the inverse of the function

$$\boldsymbol{t} \mapsto s_{\boldsymbol{q}}(\boldsymbol{t}) := \boldsymbol{t} + \frac{\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}}{c^2} \cdot \exp(\boldsymbol{t}\Omega_o) \cdot \boldsymbol{q}_o \tag{57}$$

By a convenient choice of the "origin" o we can attain that $\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}$ be orthogonal to $\Omega_o \cdot \boldsymbol{q}_o$; thus, since $\exp(\boldsymbol{t}\Omega_o) \cdot \boldsymbol{q}_o = \boldsymbol{q}_o \cos\omega\boldsymbol{t} + \frac{\Omega_o \cdot \boldsymbol{q}_o}{\omega} \sin\omega\boldsymbol{t}$, we find that

$$\boldsymbol{s}_{\boldsymbol{q}}(\boldsymbol{t}) = \boldsymbol{t} + \frac{\boldsymbol{v}_{\boldsymbol{u}\boldsymbol{u}_o}}{c^2} \cdot \boldsymbol{q}_o \cos\omega\boldsymbol{t}. \tag{58}$$

If we denote the inverse of $\boldsymbol{s_q}$ by $\boldsymbol{s} \mapsto t_{\boldsymbol{q}}(\boldsymbol{s})$, then the motion relative to the reference frame becomes, according to the LAM

$$\boldsymbol{s} \mapsto \exp(\boldsymbol{t_q}(\boldsymbol{s})\Omega_o) \cdot \boldsymbol{q}_o. \tag{59}$$

This is not a uniform rotation. Moreover, if we take a subset B_o of $\mathbf{E}$ in which $\boldsymbol{q_o}$ can vary, the corresponding world lines form a body which is rigid in the SRM but it is not rigid in the LAM.

Thus in the usual considerations of uniformly rotating rigid bodies one should specify from what point of view the body is rigid and uniformly rotating. This is important in view of the rotor Doppler shift experiments, like the Kolen-Torr experiments [24,25]. See a detailed discussion in [19,26].

We end this paper remarking that Santilli's isominkowskian spacetime [27-32] provides an alternative representation of causal events with arbitrary speeds and more particularly that his isopoincaré symmetry [28] provides the explicit form of the symmetry transformations valid for arbitrary causal speeds. The evidently expected connection between our studies and Santilli's isominkowskian spacetime will be studied at some later time.

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