

THE LORENTZ BOOST-LINK IS NOT UNIQUE.

Relative velocity as a morphism in a connected groupoid category of null objects

Zbigniew Oziewicz*

Universidad Nacional Autónoma de México

Facultad de Estudios Superiores Cuautitlán

Apartado Postal # 25

C.P. 54714 Cuautitlán Izcalli

Estado de México, México[†]

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Abstract

The isometry-link problem is to determine all isometry transformations among given pair of vectors with the condition that if these initial and final vectors coincide, the transformation-link must be identity on entire vector space. Such transformations-links are said to be the pure, or the boost, transformations. The main question, posed by van Wyk in 1986, is: how many there are pure-isometry-links among given pair of vectors of the same magnitude?

In the first part of this essay we provide the complete solution for the link problem for arbitrary isometry, for any dimension and arbitrary signature of the invertible metric tensor. We are proving that in generic case each solution of the link problem is not given uniquely by the given initial and final vectors; each solution needs the third vector called the privileged or preferred vector: the triple of vectors determine the unique pure isometry. If triple of vectors is coplanar the isometry-link is given uniquely by initial and final vectors. Non-planar systems gives, in general, infinite set of isometry-link solutions.

We apply these considerations for pure Lorentz transformations, for the Lorentz boost, parameterized by relative velocity, and we are showing that the isometric pure Lorentz transformation-link is not given uniquely by the initial and final vectors. Lorentz's boost needs a choice of the preferred time-like observer. This leads to non-uniqueness of the relative velocity among two reference systems, that was apparently not intention of Einstein in 1905.

In order to have axiomatically the unique relative velocity among pair of massive bodies, we propose a connected groupoid category of massive bodies in mutual motions. Groupoid is a small category in which all morphisms are isomorphisms. In groupoid category of massive bodies, each morphism is the unique relative velocity (not non-unique isometric Lorentz transformation). We propose to consider the relative velocity as the primary concept with dichotomy: the unique velocity-morphism, *versus* the morphisms given by a set of non-unique isometric Lorentz boost-links.

Presently, as during the XX century, the Lorentz covariance is the cornerstone of physical theory. Our main conclusion is: observer-dependence (and -independence), and the Lorentz-covariance (and invariance), are different concepts. The same statement holds in physics with absolute simultaneity: XVII-century observer-independence is not the same as the XX-century Galilean-group-invariance-covariance. We think that Lorentz covariance will dwindle in importance.

The kinematics of groupoid-category-relativity is ruled by Frobenius algebra, whereas the dy-

namics needs the Frölicher-Richardson non-associative algebra.

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[†]Electronic address: oziewicz@servidor.unam.mx

I. RELATIVITY CONCEPTS in terms of CATEGORY THEORY

Science is the meaning of reality, not the photograph of reality.

H. Bergman 1929, cited in *Boston Logical Studies* [1974, p. 400].

Relativity of the space, *i.e.* many relative spaces of Galileo [Galileo 1632], and relativity of time, *i.e.* many relative simultaneity of Albert Einstein [Einstein 1905], and many relative proper-times of Minkowski [Minkowski 1908], all were formulated long before the first concepts of the category theory were invented in 1945 [Eilenberg and Mac Lane 1945].

The relativity principle was known since Kopernicus (Copernicus) [1543] and Galileo [1632]. Poincaré [1904], Lorentz [1904] and Einstein [1905], formulated the relativity principle in the following form: laws of physics, including electromagnetic phenomena, should be independent of the choice of the inertial system of reference, should be inertial-observer-free. This principle is said to be the principle of the special (inertial) relativity, with constant relative velocities. Einstein extended this to the principle of general relativity, stating that laws of physics, including gravity, should be observer-free relative to *all* reference systems, including all non-inertial systems, accelerated, deformed, rotated.

Brillouin stressed in [1970] that the reference system must be physical *massive* body, and not mathematical coordinate system. The *massless* cosmic microwave background radiation, often considered as the ‘natural preferred frame of reference’, see e.g. [Braxmaier et al. 2002], can not be considered as the physical observer.

The crucial is the mathematical model of the physical massive body.

The physical massive body, with mass distribution, is considered to be located in the primitive absolute configuration space. Lawvere [2002], in the science of materials, consider the topos category of spaces, with relations between body and absolute space.

Wagh [2006] propose universal relativity that is going beyond the gravity, and for all reference systems, in framework of point-free topology. It is proposed that a physical reference system is given by a complete lattice, called *frame*, and it is noted that category of such physical bodies/frames is not a topos category.

The above attempts of describing the physical massive bodies in a topos category, or in non-topos category, takes as the primitive concepts the space, location, topology, distance. For another approach see also [Laudal 2005].

Contrary to the above philosophy, here we propose to consider the relative velocity as the most primitive concept. The relative velocity, not necessarily constant, being the categorical morphism in a connected groupoid category of massive physical bodies. Groupoid is a small category in which all morphisms (in our case all relative velocity-morphisms) are isomorphisms. In this philosophy, the relative spaces and relative (proper)-times are derived secondary concepts.

A. Overlook

In the Einstein special relativity theory an object is an observer that is understood just as a reference frame (a basis) or as the coordinate-system. A morphism among such objects is the isometric inhomogeneous Lorentz coordinate transformation. A relative velocity is reciprocal (skew-symmetric), and is parameterizing the isometric Lorentz-boost among two bodies. In this essay we are going to show that such 'isometric' relative velocity is preferred-laboratory-dependent, see (IV.8)-(VI.6), and the addition of such *ternary* velocities must be non-associative.

In the Lorentz-group-free categorical relativity theory, proposed here, an observer is (1+3)-split-idempotent, and a morphism between observers is an intrinsic unique binary relative velocity-morphism, that is not skew-symmetric (not reciprocal). This implies that the addition of binary velocities must be associative, but a change of an observer-idempotent is not an isometry (not Lorentz transformation).

Albert Einstein in 1905 made reciprocity assumption that the inverse of the addition of relative velocities must be the same as for the Galilean absolute simultaneity. We are going to show in this essay that the reciprocity axiom, that the mutual speeds of two reference systems differ by sign solely, leads to ternary relative velocity (with respect to the preferred time-like (massive) reference system), with non-associative addition. We think that the reciprocity axiom belongs to absolute-simultaneity physics and is not needed by relativity theory.

The central notion of our new proposal is an enriched groupoid category of most general non-inertial observers (time-like vector fields), and an operator algebra generated by (1+3)-splits-idempotents, that hopefully is a Frobenius algebra. Thus the primary notion is not a space-time, but an operator algebra, in some analogy to formulation of quantum mechanics

outside of the Hilbert-space model. This is background independent formulation, without a fixed background spacetime. A spacetime is seen as a modul/comodule over this primary operator algebra.

Starting from the associative addition of binary relative velocities-morphisms, and selecting the preferred massive reference system, one can derive the non-associative addition of ternary Lorentz-boost-velocities. This derivation will be published elsewhere.

The different relative velocities (the unique binary velocity-morphisms in a groupoid category of observers, versus many ternary Lorentz-boost-links) and different additions (the Lorentz-group-free associative addition, *versus* Lorentz-boost-based non-associative addition) are related by a map (in Frobenius algebra) that do not extend to the Lorentz transformation.

The main conclusion: the observer-independence and the Lorentz-group-invariance must not be identified as was believed during XX century.

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The present essay is inspired also by hot and long email-discussion we have had in 2004 with David Finkelstein (Georgia Institute of Technology) and with Abraham Ungar (North Dakota State University), on uniqueness of the Lorentz boost, and on the meaning of the Lorentz relativity transformation. This subject is presented as coordinate-dependent and basis-dependent matrix calculus in Ungar's monograph [Ungar 2001]. Instead I was attempting to be basis-free and coordinate-free. I am thankful to Abraham Ungar and to David Finkelstein for this hot email correspondence.

I like to thank Bernard Jancewicz (Uniwersytet Wrocławski, Poland), and to Professor Sanjay M. Wagh (Central India Research Institute, Nagpur), for inspiring correspondence and for pointing important references.

II. ISOMETRY IS COORDINATE-FREE AND BASIS FREE

The Lorentz relativity transformation is an *isometry*. In the last Sections of this essay we propose an alternative mathematical relativity theory that does not need the Lorentz isometric transformations. The alternative formulation we call the Lorentz-group-free relativity. It is the groupoid category of reference systems, where morphisms are not isometries. In order to compare these two mathematical formulations of relativity theory, one with the group of Lorentz relativity transformations, and another without of the group of the Lorentz relativity transformations, we wish to clarify what is the coordinate-free and basis-free isometry. Now, in XXI century, the majority of the physics community are not familiar with the coordinate-free discussion. A tensor as a coordinate-free concept is alien, and coordinate-free reasoning is considered as the defect ‘enveloping the physical ideas’. Nowadays, year 2006, the physics community demand the presentation explicitly in ‘sacred’ coordinates and basis-dependent matrices of XVIII and XIX centuries. Coordinates and matrices (matrix of the boost, Dirac matrix, etc) is considered as the only legitimate language of physics, maybe the language for all science? Contrary to this mainstream adoration of coordinates and bases (= frames), here we wish explain that the concept of an isometry is coordinate-free and basis-free. We believe that coordinates and/or bases introduce something irrelevant into scientific thinking, and obscure both, the mathematical and the physical ideas. For discussion of the obscure meaning of the coordinate systems we refer to [Hermann Weyl 1921, 1926].

Each Lorentz and Poincaré (inhomogeneous Lorentz) transformation is an isometry. The Lorentz and Poincaré groups are the groups of symmetries of the *metric-tensor* of the *empty*-space-time, no matter. The question is: why relativity transformation from one *massive* reference system to another massive reference system must be obligatory the isometry of the *empty*-energy-less spacetime?

In order to understand the physical meaning of the Lorentz relativity transformation (also viewed as coordinate-free), and therefore the physical meaning and the interpretation of the Einstein’s special relativity theory, it is desirable to understand the mathematical concept of isometry outside of the relativity theory.

So, what is an isometry? Consider a module over associative commutative algebra, or a vector space V over a field, of arbitrary finite dimensionality that is irrelevant for the

discussion in this Section. Consider for a simplicity a vector space, \mathbb{R} -space V , no relativity theory yet. Let V^* be a dual \mathbb{R} -space, and let $g \in V^* \otimes_{\mathbb{R}} V^*$, be symmetric metric tensor, considered as the map $V \otimes V \rightarrow \mathbb{R}$, and as the isomorphism, $g = g^* : V \rightarrow V^*$. The signature of g is arbitrary and irrelevant for the discussion in the present Section. Let $A, B \in V$ be vectors, then the scalar product is usually denoted by $A \cdot B = g(A, B) \in \mathbb{R}$. However, in fact this means $g(A \otimes B)$, because the domain of metric *tensor* g are second rank tensors, like $A \otimes B$, and not Cartesian pairs $(A, B) \in V \times V$. We will use $A \cdot B$ for brevity. The scalar product g of vectors, $g : V \rightarrow V^*$, extends by algebra homomorphism to scalar product g^{\otimes} of all tensors, $g^{\otimes} : V^{\otimes} \rightarrow V^{*\otimes}$, and extends to scalar product g^{\wedge} of all Grassmann multi-vectors, $g^{\wedge} : V^{\wedge} \rightarrow V^{*\wedge}$.

II.1 Example. Let $A \wedge B$ and $P \wedge Q$ be simple bi-vectors in $V^{\wedge 2} \equiv V \wedge V$. Then

$$(A \wedge B) \cdot (P \wedge Q) \equiv g^{\wedge}((A \wedge B) \otimes (P \wedge Q)) = (A \cdot P)(B \cdot Q) - (A \cdot Q)(B \cdot P).$$

II.2 Definition (Isometry). An endomorphism L of \mathbb{R} -space V , $L \in \text{End } V = V \otimes V^*$, is said to be g -isometry if the scalar product is invariant,

$$\forall A, B \in V, \quad g((LA) \otimes (LB)) = (LA) \cdot (LB) = A \cdot B, \quad (\text{II.1})$$

$$L^* \circ g \circ L = g. \quad (\text{II.2})$$

The set of all g -isometries is a Lie group, denoted by O_g ,

$$\dim V = n \quad \implies \quad \dim(O_g) = \dim(V^{\wedge 2}) = \binom{n}{2} = \frac{n(n-1)}{2}. \quad (\text{II.3})$$

If $P \in V$ is a vector (basis-free!), and $\alpha \in V^*$ is a covector, then $P \otimes \alpha \in \text{End } V$, is said to be simple endomorphism, with quadratic minimal polynomial (kind of idempotency),

$$\text{tr}(P \otimes \alpha) = \alpha P, \quad (P \otimes \alpha)^2 = (\alpha P)(P \otimes \alpha), \quad (\text{II.4})$$

$$g \circ (P \otimes \alpha) = \{(gP) \otimes (g^{-1}\alpha)\} \circ g. \quad (\text{II.5})$$

In this essay we are interested in the parametrization of isometry in terms of basis-free vectors, and in terms basis-free Grassmann bivectors, keeping in mind the application for the parametrization of the Lorentz boost in terms of ‘the vector of relative velocity’ that will be presented in the basis-free way in the next Sections.

One can try parameterize isometry $L \in O_g$ in terms of the single vector P as,

$$L_P = \text{id} - P \otimes \alpha, \quad (\text{II.6})$$

for unknown covector α . Inserting this expression into the definition of the isometry (II.2), we get the unique solution for covector α , provided that $P^2 \neq 0$. Such single-vector-parameterized-isometry appears to be the ray-parameterizations that depends on one-dimensional subspace span by non light-like vector field (or a vector) P . Moreover, such isometry must be unipotent = reflection,

$$L_P = \text{id} - 2 \frac{P \otimes gP}{P^2} \implies (L_P)^{-1} = L_P. \quad (\text{II.7})$$

The next step is to look for the parametrization of isometry in terms of the pair of vectors, the triple of vectors \dots , etc, *i.e.* to consider the sum of the simple endomorphism $L = \text{id} - \sum_i (P_i \otimes \alpha^i)$. The sum of simple endomorphism (simple tensors) that is not simple is said to be an entanglement. For more details about entanglement, see the Lecture by Guillermo Morales-Luna in this Volume.

Consider the parametrization of isometry in terms of the sum of two idempotents. This is the following ansatz for two unknown covectors, α and β , or for four unknown scalars, a, b, c, e , as was considered by van Wyk in 1958,

$$\begin{aligned} L &= \text{id} - P \otimes \alpha - Q \otimes \beta, & P \wedge Q \neq 0, & \quad \alpha \wedge \beta \neq 0, \\ g^{-1}\alpha &= aP + bQ, & g^{-1}\beta &= cP + eQ. \end{aligned} \quad (\text{II.8})$$

Inserting above ansatz into (II.1)-(II.2) we get the following system of three scalar equations

$$(aP + cQ)^2 = 2a, \quad (bP + eQ)^2 = 2e \quad ((a + b)P + (c + e)Q)^2 = 2(a + b + c + e). \quad (\text{II.9})$$

One can solve above system (II.9) in terms of the next ansatz in terms of one unknown P - and Q -dependent scalar field $\gamma \equiv \gamma_{P,Q}$,

$$a = \frac{Q^2}{\gamma + 1}, \quad b = -1 - \frac{P \cdot Q}{\gamma + 1}, \quad e = \frac{P^2}{\gamma + 1}, \quad c = +1 - \frac{P \cdot Q}{\gamma + 1}. \quad (\text{II.10})$$

Then, the isometry condition (II.1)-(II.2) and the system of equations (II.9)-(II.10) is reduced to the following single equation for a scalar $\gamma_{P \wedge Q}$,

$$(\gamma_{P \wedge Q})^2 = 1 - (P \wedge Q)^2, \quad (P \wedge Q)^2 \leq 1. \quad (\text{II.11})$$

The last condition on the magnitude of the bivector $P \wedge Q$ assure that the scalar field $\gamma_{P \wedge Q}$ is real. Isometry parameterized by the pair of vectors, $\{P, Q\}$, (II.8)-(II.11) possess the third order minimal polynomial (distinct from reflection (II.7)), and this minimal polynomial depends, through (II.11), on the magnitude of the bivector $(P \wedge Q)^2$ only,

$$(L - 1) \{L^2 + 2(\gamma - 2)(L + 1)\} = 0. \quad (\text{II.12})$$

The parametrization (II.8) in terms of the sum of two simple endomorphisms, is in fact the simple-bi-vector parametrization

$$V^{\wedge 2} \ni P \wedge Q \quad \leftrightarrow \quad L_{P \wedge Q} \in O_g. \quad (\text{II.13})$$

This can be seen as follows. The vector space of Grassmann bi-vectors $V^{\wedge 2}$, inside of the Clifford algebra, $V^{\wedge 2} \subset \mathcal{Cl}(V, g)$, is the Lie algebra of the Lie group O_g , $\dim(O_g) = \dim(V^{\wedge 2})$. There is the Lie algebra morphism $M : V \wedge V \longrightarrow V \otimes V^*$, that we are going to describe now. For each vector $v \in V$, $gv \in V^*$, we denote by $i_{gv} \in \text{der}(\text{Grass})$ the graded derivation of the Grassmann algebra V^\wedge . For each bivector $b \in V^{\wedge 2}$ (not necessarily simple), and for each vector $v \in V$, the linear map M is defined as follows

$$V^{\wedge 2} \ni b \quad \longrightarrow \quad M_b v \equiv -i_{gv} b \quad \in V. \quad (\text{II.14})$$

One can show that the image of the above map, $\text{im } M \subset \text{End } V$, are trace-less and g -skew-symmetric endomorphisms. Therefore M extends to the Lie algebra isomorphism from the Lie algebra of bivectors \mathfrak{o}_g , to the Lie algebra of trace-less g -skew-symmetric endomorphism of V .

This means that an isometry $L \in O_g$ can be generated by bi-vectors,

$$M_{P \wedge Q} \equiv P \otimes gQ - Q \otimes gP \quad \in \text{End } V. \quad (\text{II.15})$$

The endomorphism $M_{P \wedge Q}$ possess the third order minimal polynomial, and therefore there is no surprise that the following binomial coincide with the parametrization (II.8)-(II.11),

$$L_{P \wedge Q} = \text{id} + M_{P \wedge Q} + \frac{1}{\gamma + 1} (M_{P \wedge Q})^2. \quad (\text{II.16})$$

In this way we proved the following theorem, that holds for $2 \leq \dim V$, and for arbitrary signature of the metric tensor g .

II.3 Theorem. *Let the simple bivector $P \wedge Q$ satisfy the following condition (II.17), and the scalar λ is given as follows*

$$(P \wedge Q)^2 \leq 1, \quad \gamma_{P \wedge Q} \equiv \sqrt{1 - (P \wedge Q)^2}. \quad (\text{II.17})$$

Let an endomorphism $L_{P \wedge Q} \in \text{End } V$, be defined as follows

$$\begin{aligned} L_{P \wedge Q} &= \text{id} + P \otimes gQ - Q \otimes gP + \frac{1}{\gamma + 1} (P \otimes gQ - Q \otimes gP)^2 \\ &= \text{id} - \frac{1}{\gamma + 1} P \otimes \{Q^2 gP - (P \cdot Q + \gamma + 1)gQ\} - \frac{1}{\gamma + 1} Q \otimes \{P^2 gQ - (P \cdot Q - \gamma + 1)gP\}. \end{aligned} \quad (\text{II.18})$$

Then, the above endomorphism (II.18) is the g -isometry, $L_{P \wedge Q} \in O_g$, with the minimal polynomial (II.12). An inverse is $\{L_{P \wedge Q}\}^{-1} = L_{Q \wedge P}$.

Proof. It is sufficient to verify the following equalities for all vectors K, N ,

$$L_{P \wedge Q} \circ L_{Q \wedge P} = \text{id}, \quad (L_{P \wedge Q} K) \cdot (L_{P \wedge Q} N) = K \cdot N. \quad (\text{II.19})$$

One must use condition (II.17). \square

The discussion which follows in the next Sections will be based on the technical knowledge of the above coordinate-free and basis-free simple-bivector-parametrization of g -isometry (II.18) with condition (II.17). The isometry $L_{P \wedge Q} \in O_g$ (II.18) is a second order polynomial in generating bivector from the Lie algebra $P \wedge Q \in \mathfrak{o}_g$, (II.16). In explicit expression for the isometry $L_{P \wedge Q}$, (II.18), we see the vectors P and Q as separated, and the bivector $P \wedge Q$, as the *only* intrinsic variable, see (II.16), is hidden.

The g -isometry (II.18), imply in particular that,

$$L_{P \wedge Q} P = \left\{ 1 + P \cdot Q - \frac{1}{\gamma + 1} (P \wedge Q)^2 \right\} P - P^2 Q, \quad (\text{II.20})$$

$$L_{P \wedge Q} Q = \left\{ 1 - P \cdot Q - \frac{1}{\gamma + 1} (P \wedge Q)^2 \right\} Q + Q^2 P. \quad (\text{II.21})$$

Let $P^2 Q^2 = +1$. Then in (II.17), $\gamma = |P \cdot Q|$,

$$L_{P \wedge Q} P = \begin{cases} 2(P \cdot Q)P - P^2 Q & \text{if } 0 \leq P \cdot Q, \\ -P^2 Q & \text{if } P \cdot Q \leq 0 \end{cases} \quad (\text{II.22})$$

$$L_{P \wedge Q} Q = \begin{cases} Q^2 P & \text{if } 0 \leq P \cdot Q, \\ -2(P \cdot Q)Q + Q^2 P & \text{if } P \cdot Q \leq 0 \end{cases} \quad (\text{II.23})$$

Ivanitskaja consider the Lorentz group $O(1, 3)$, and ansatz in terms of not simple bivectors, similar to, $L = \text{id} - \sum_{i=1}^{i=4} (P_i \otimes \alpha^i)$, [Ivanitskaja 1979, Chapter VI, page 292, formula (26.15)].

A. Bivector has many presentations

Each simple Grassmann bivector is $SL2$ -invariant, *i.e.* has many $SL2$ -presentations in terms of vectors. For scalars a, b, c, e , $\begin{pmatrix} a & b \\ c & e \end{pmatrix} \in SL2$, we have,

$$ae - bc = 1 \iff (aP + bQ) \wedge (cP + eQ) = P \wedge Q. \quad (\text{II.24})$$

We need the following idempotent for non light-like vector $P^2 \neq 0$,

$$p \equiv \frac{P \otimes gP}{P^2} \implies p^2 = p, \quad \text{tr } p = 1. \quad (\text{II.25})$$

The above freedom of the presentation (II.24), can be used to fulfill the orthogonality of vectors,

$$P \wedge Q = P \wedge (Q - fP), \quad P \cdot (Q - fP) = 0 \implies f = \frac{P \cdot Q}{P^2}, \quad (\text{II.26})$$

$$W \equiv (\text{id} - p)Q = Q - \frac{P \cdot Q}{P^2}P, \quad \text{is orthogonal to } P, \quad W \cdot P = 0. \quad (\text{II.27})$$

In what follows it is convenient to use different presentations of the bivectors,

$$P \wedge Q = P \wedge (\text{id} - p)Q = P \wedge W, \quad \text{where } P^2 \neq 0 \quad \text{and} \quad P \cdot W = 0. \quad (\text{II.28})$$

We stress that the change of the presentation in terms of different vectors does not change the bivector in question, and does not change the isometry, $L_{P \wedge Q} = L_{P \wedge W}$. This is the change of the parametrization of the same isometry. In the orthogonal presentation, $P \cdot W = 0$, of the simple bivector $P \wedge W$, we have

$$\begin{aligned} (P \wedge W)^2 &= P^2 W^2 \leq 1, \quad \gamma = \sqrt{1 - P^2 W^2}, \\ L_{P \wedge W} &= \text{id} + P \otimes gW - W \otimes gP - \frac{1}{\gamma + 1} (W^2 P \otimes gP + P^2 W \otimes gW), \\ L_{P \wedge W} P &= -P^2 W + \left(1 - \frac{P^2 W^2}{\gamma + 1}\right) P, \\ L_{P \wedge W} W &= +W^2 P + \left(1 - \frac{P^2 W^2}{\gamma + 1}\right) W. \end{aligned} \quad (\text{II.29})$$

The bivector is presentation-independent, therefore the magnitude of the bivector is presentation-independent also.

III. ISOMETRY AS A LINK: THE COMPLETE SOLUTION

In this Section we consider the link equation for the isometry. This is a mathematical problem outside of relativity theory, and is formulated for dimension ≥ 2 , and for arbitrary signature of the invertible metric tensor field g .

There is an obvious fact in group theory, emphasized by Wigner [1939], that every group transformation from an initial source-vector R , to a target-vector S , is up to the Wigner little subgroups of R and S , known also as stabilizers, the stability subgroups. For example, the Wigner little subgroup of the isometry group O_g of a vector R is defined as follows

$$O^R \equiv \{k \in O_g, kR = R\}. \quad (\text{III.1})$$

Therefore the link equation $LR = S$ for the given vectors R and S , such that

$$(R - S) \cdot (R + S) = R^2 - S^2 = 0, \quad (\text{III.2})$$

and for unknown isometry $L \in O_g$, has the little-group-ambiguity,

$$\forall r \in O^R \quad \text{and} \quad \forall s \in O^S, \quad LR = (s \circ L \circ r)R = S. \quad (\text{III.3})$$

If an isometry $L = L(R, S)$ solve the link equation $LR = S$, then also $(s \circ L \circ r)$ is a solution, and there is infinite set of links up to the Wigner-stabilizer ambiguity. Such Wigner-little-group non-uniqueness means that the solution of link-equation $LR = R$ need not to be an identity for $R = S$, $L(R, R) \neq \text{id} \in \text{End } V$. In order to be stabilizer-independent we restrict the solutions of the link equation $LR = S$, by pure-link condition, $L(R, R) = \text{id}$, $L|(R = S) = \text{id}$.

III.1 Definition (Link equation and boost). Let R and S be given vectors such that $R^2 = S^2$, *i.e.* $(R - S) \cdot (R + S) = 0$. The link equation for the unknown g -isometry $L \in O_g$, is $LR = S$. The isometry-link solution $L = L(R, S)$, such that, $L(R, R) = \text{id}$, is said to be a pure-isometry-link, or a boost.

The question is about the uniqueness of the isometry-link: how many there are the different pure-isometry-links, $L = L(R, S) \in O_g$, for the given vectors R and S , such that $(R - S) \cdot (R + S) = R^2 - S^2 = 0$?

When an isometry $L \in O_g$ is a reflection, $L^2 = \text{id}$, parameterized by a single unknown non-zero vector P , the link equation $L_P R = S$, for given conditions, $(R - S) \cdot (R + S) = 0$

and $(R - S)^2 \neq 0$, has solution up to non-zero scalar $\mu \neq 0$, $P = \mu(R - S)$. This ‘Galilean subtraction’, $R - S$, can be interpreted as a ‘relative parametria from R to S ’,

$$L_P R = S \iff P = \mu(R - S). \quad (\text{III.4})$$

Our problem in this Section is to find the most general solution for the link equation for g -isometry parameterized by simple-bivector (II.18), $L_{P \wedge Q} R = S$. This is a problem to find *all* simple bivectors $P \wedge Q$, for the given pair of vectors R and S , subject to constraint $(R - S)(R + S) = 0$. This problem has the separate complete solution for the generic case when $(R - S)^2 \neq 0$, and another solution for singular case $(R - S)^2 = 0$.

III.2 Assumption. In what follows we assume that the system of three non-zero vectors $\{R, S, P\}$, is subject to the following conditions,

$$\begin{aligned} (R - S) \cdot (R + S) &= 0, \quad \text{and} \quad (R - S)^2 \neq 0, \\ P \cdot (R + S) &\neq 0, \quad \text{and} \quad P \wedge (R - S) \neq 0, \\ \{P \wedge (R - S)\}^2 + \{P \cdot (R + S)\}^2 &= P^2(R - S)^2 + 4(P \cdot R)(P \cdot S) \neq 0. \end{aligned} \quad (\text{III.5})$$

The non-zero scalar $\mu = \mu(P, R \pm S)$, for the above system of three vectors is defined as follows

$$\mu \equiv \frac{2P \cdot (R + S)}{\{P \wedge (R - S)\}^2 + \{P \cdot (R + S)\}^2}. \quad (\text{III.6})$$

Then, a vector μP is homogeneous in P , depends on one-dimensional ray span by non-zero vector P , denoted by P -ray.

III.3 Theorem (Main: isometry-links). *Consider the isometry $L \in O_g$ parameterized by simple bivector (II.18). Then, each solution of the isometry-link equation, $LR = S$, for the given vectors R and S , is given by a P -ray, also referred to as the privileged (exterior) or preferred P -ray, subject to conditions (III.5)-(III.6). All isometry-links from R to S , are parameterized by variable rays $\{\mu P\}$, and have the following explicit form,*

$$\begin{aligned} L_{(\mu P) \wedge (R - S)} = \text{id} - \frac{2P \otimes \{(R - S)^2 g P - 2(P \cdot R)g(R - S)\}}{P^2(R - S)^2 + 4(P \cdot R)(P \cdot S)} \\ - \frac{(R - S) \otimes \{2P^2 g(R - S) + 4(P \cdot S)gP\}}{P^2(R - S)^2 + 4(P \cdot R)(P \cdot S)}, \end{aligned} \quad (\text{III.7})$$

$$L_{(\mu P) \wedge (R - S)} R = S, \quad \gamma + 1 = \mu P \cdot (R + S), \quad (\text{III.8})$$

$$L_{(\mu P) \wedge (R - S)} S = 2(\mu P \cdot S)S + (1 - 2\mu P \cdot S)R - (R - S)^2 \mu P. \quad (\text{III.9})$$

Proof. Let unknown isometry for unknown simple bivector $P \wedge Q$, is in the form (II.8)-(II.18),

$$L_{P \wedge Q} = \text{id} - P \otimes \alpha - Q \otimes \beta,$$

$$(\gamma + 1)\alpha \equiv Q^2 gP - (P \cdot Q + \gamma + 1)gQ, \quad (\gamma + 1)\beta \equiv P^2 gQ - (P \cdot Q - \gamma + 1)gP. \quad (\text{III.10})$$

The link equation, $LR = S$, imply the vanishing of the tri-vector

$$(P \wedge Q) \wedge (R - S) = 0 \iff P \wedge Q = (\mu P) \wedge (R - S) \neq 0. \quad (\text{III.11})$$

Inserting an ansatz $Q = \mu(R - S)$ into link equation $LR = S$, we get $\alpha R = 0$ and $\mu\beta R = 1$. Therefore (III.10) imply

$$\alpha R = 0 \iff \gamma + 1 = \mu P(R + S), \quad (\text{III.12})$$

$$\mu\beta R = 1 \iff \mu = (\text{III.6}). \quad (\text{III.13})$$

In order to finish the proof we must check that the above solution is compatible with the necessary condition (II.11).

Instead of the above deduction of (III.7), Reader can check directly that for this explicit isometry P -link (III.7), the identities (III.8)-(III.9), holds.

The isometry-link (III.7) is the most general complete solution subject to Assumption III.2. □

Note that in the above P -link, a simple bivector, $(\mu P) \wedge (R - S)$, is up-to P -ray μP , in some analogy to the link-solution for single vector (III.4), $\mu(R - S)$, that is up to arbitrary non-zero scalar μ .

III.4 Noteworthy (Non-uniqueness of isometry-link). The link equation is crucial for the interpretation of the isometry $L = L(R, S)$ in terms of the known initial vector R and known final vector S . In the case of the Lorentz transformations in the special relativity, the Lorentz-link problem was considered by Donald Fahnline [Fahnline 1982, Section III], and by van Wyk in 1986. We will consider the specific pure Lorentz transformations in details in the next Section.

The non-uniqueness of the isometry-link (III.7) is important for understanding the isometry, and for understanding the Einstein special relativity. We can say that (III.7)-(III.8) is the link from R to S as *seen* by ‘a privileged = preferred’ vector P . We abbreviate this isometry-link (III.7) briefly by P -link.

For *each* vector P subject to conditions (III.5), this isometry- P -link is a ‘pure’ transformation, in the meaning used for pure Lorentz transformations in special relativity, *i.e.* for transformations that do not involve a spatial rotation. P -dependent isometry-link $L_{(\mu P)\wedge(R-S)}$, (III.7), for $R = S$ is an identity, $L|(R = S) = \text{id}$. The non-uniqueness of the isometry-link due to (μP) -dependence, (III.7), has nothing to do with Wigner’s little groups.

This technical non-uniqueness is equivalent to non-uniqueness of the group embedding $O(3) \hookrightarrow O(1, 3)$.

We are going to demonstrate in the next Sections, and in more details in the separate publication, that exactly the preferred-vector-dependence of the isometry-link, *i.e.* non-uniqueness, is the primary source of the non-associativity of the velocity addition in special relativity [Ungar 1990, 2001], the primary source of the Thomas precession = rotation = gyration [Fisher 1972; Ungar 1988, 1989, 1991, 1997, 2001; Urbantke 1990], the primary source that the relativistic dynamics of many-body system in special relativity can not be resolved, the primary source of some problems in electromagnetism [Valent 2002], the primary source of some paradoxes of the special relativity, e.g. [Sastry 1987], including the Mocanu paradox [1986], and the primary source that the special relativity is miss-understood, etc.

The most important conclusion at this point is that the above bivector-parametrization of isometry, (II.12)-(II.18)-(II.29)-(III.7), holds for dimension no less than two, and for arbitrary signature of the metric tensor.

III.5 Theorem. *The link equation $L_{P\wedge(R-S)}R = S$ for the case when $(R - S)^2 = 0$, for unknown vector field P and scalar field γ is reduced to the following set of two scalar conditions*

$$(\gamma + 1)(P \cdot R - 1) = (P \cdot R)(P \cdot R - P \cdot S), \quad \gamma^2 = 1 + (P \cdot R - P \cdot S)^2. \quad (\text{III.14})$$

IV. PLANAR SYSTEM OF THREE VECTORS

IV.1 Definition (Planar system). A system of three vector fields $\{P, R, S\}$, is said to be planar if vanishes the tri-vector field,

$$P \wedge R \wedge S = 0. \quad (\text{IV.1})$$

The cases of planar and no-planar preferred-vector P , *relative* to the given plane $R \wedge S$, *i.e.* relative to the initial R and final S vectors, are essentially different, and deserve the separate consideration.

From now on, the vectors R and S are such that $R^2 = S^2 \neq 0$. Then we have idempotents (II.25),

$$r \equiv \frac{R \otimes gR}{R^2} \quad \text{and} \quad s \equiv \frac{S \otimes gS}{S^2}, \quad (\text{id} - r)R = 0. \quad (\text{IV.2})$$

IV.2 Lemma (Planar ternary system). *Let $R^2 = S^2 \neq 0$ and $(S + R)^2 \neq 0$. Then*

$$\{(\mu P) \wedge (R - S)\}_{|P \wedge R \wedge S = 0} = \frac{S \wedge R}{S^2}. \quad (\text{IV.3})$$

When in Theorem III.3, a preferred P -ray is coplanar relative to the given plane $R \wedge S$, then the bivector generating the pure-isometry-link is given uniquely in terms of initial and final vectors.

The planar g -isometry-link, $L_{\frac{S \wedge R}{S^2}}$, from R to S , has the following expression

$$L_{\frac{S \wedge R}{R^2}} = \text{id} - 2 \frac{(R + S) \otimes g(R + S)}{(R + S)^2} + 2 \frac{S \otimes R}{S^2}, \quad (\text{IV.4})$$

$$L_{\frac{S \wedge R}{R^2}} R = S \quad \text{and} \quad L_{\frac{S \wedge R}{R^2}} S = (2s - \text{id})R. \quad (\text{IV.5})$$

The planar g -isometry-link (IV.4)-(IV.5), generalize for any dimensions ≥ 2 , and arbitrary signature, the isometry derived by Fahnline [Fahnline 1982, Section III, formulas (15)-(16)-(18)], see (VI.1).

The link-equation $LR = S$ together with the extra condition (IV.5), is equivalent to the Matolcsi definition [Matolcsi 1994, §1.3.8], and fixes the preferred P -ray in (III.7)-(III.9), uniquely to be planar,

$$\left. \begin{array}{l} LR = S \\ LS = (2s - \text{id})R \end{array} \right\} \iff P \wedge R \wedge S = 0. \quad (\text{IV.6})$$

It is true that the isometry $L_{P \wedge Q}$ is uniquely given by bivector $P \wedge Q$. However it is *not* true that the g -isometry-link, $L_{P \wedge Q}$, is uniquely given by initial R and final S vectors, as could be suggested e.g. in [Ungar 2001, page 348, Theorem 11.16]. The bivector $P \wedge Q$, generating the isometry, is not uniquely given by initial R and final S vectors! To get unique isometry-link one need to chose a preferred P -ray (III.5), or equivalently, put some

extra condition on isometry, like the Fahnline condition (IV.5) [Fahnline 1982, Section III], assumed explicitly also by Matolcsi [1994, §1.3.8], and assumed implicitly in [Ungar 1988, 2001].

Each P -link from R to S , is given in terms of the simple bivector, and therefore each must have the same third order minimal polynomial (II.12) uniquely determined by the scalar magnitude of the bivector. Therefore one-way to see for the given pair of vectors, initial R and final S , that non-planar P -link (III.7) is essentially different from planar-link (IV.4), is to verify the difference of the scalar magnitudes

$$P \wedge R \wedge S \neq 0 \iff \{(\mu P) \wedge (R - S)\}^2 \neq \frac{(S \wedge R)^2}{S^4} \quad (\text{IV.7})$$

The planar-link (IV.4), gives the link-identity (IV.5). However we must avoid incorrect impression of the uniqueness of the Lorentz-link, and not forget about non-planar solutions of the link-equation (III.9), for dimensions ≥ 3 .

The planar isometry-link, $R \mapsto S$, is generated by bivector $\frac{S \wedge R}{S^2 = R^2}$, however must not be tempted to be defined as the *only* unique isometry-link. Compare for example how this subject is presented when specialized for the four-dimensional Minkowski space-time with the metric tensor of Lorentzian signature. Among many other we refer to [Fahnline 1982, Section III, formulas (15)-(16)-(18); Matolcsi 1994, §1.3.8; Ungar 2001, Theorem 11.16 on page 348 does not holds for non-planar preferred observer; Urbantke 2003]. Fahnline [1982 Section III] gives isometry-link (IV.4)-(IV.5) for irrelevant ‘inertial frame’=basis, and comment that ‘there are other (pure) Lorentz transformations’-links from R to S (the other are non-planar).

The happy conclusion, that the pure isometry-link, $R \rightarrow S$, is *uniquely* given by initial R and final S vectors, [Ungar, Theorem 11.16 on page 348], is incorrect conclusion for dimensions ≥ 3 . Such conclusion ignore the multitude of non-planar g -isometry-links from initial R to final S given by (III.7). To have just one Lorentz boost from R to S you need made *a choice* of a preferred P -ray (III.5), to be planar or non-planar, with many possibilities for non-planarity relative to the given plane $R \wedge S$.

The choice of P -ray is equivalent to a *choice* of an embedding $O(3) \hookrightarrow O(1,3)$.

A. Why the uniqueness of the boost is so much desired?

Let me comment at this moment about Fahnline's expression for the isometry-link (IV.4)-(IV.5), that assume implicitly the planar preferred-ray. Fahnline arrived to his expression starting from the pure Lorentz transformation of coordinates, parameterized by 'ordinary relative velocity', following Einstein [1905] (we consider this in Sections VI-VII). The Fahnline planar isometry-link is equivalent to Einstein's innocent coordinate transformation, that hide the physical meaning of the 'ordinary velocity' among reference systems. The pure Lorentz transformation are parameterized by 'ordinary relative velocity' only, having the most important physical interpretation, and therefore, since 1905, there is absolute certainty, dogma, about one-to-one correspondence among relative velocity, that must be *unique* among two reference systems, and the pure Lorentz transformations-links. Shortly: the unique velocity of reference system S relative to reference system R , must imply the unique Fahnline's isometry-link (IV.4). Many different non-planar preferred-ray-dependent pure Lorentz transformations-*links* from R to S , as given by (III.7), would destroy happiness because imply non-unique relative velocity among reference systems, that was *not* Einstein's intention in 1905. This is, I believe, the psychological reasons that during XX century the physics community accepted the Einstein coordinate transformations, equivalent to Fahnline's *planar* pure Lorentz transformation-link (IV.4)-(IV.5)), as the *unique* boost. This ignore Theorem III.3, giving the infinite family of non-planar pure Lorentz transformations-links, with infinite family of non-planar relative isometric velocities among two bodies.

The isometric velocity, parameterizing the pure Lorentz transformation, needs, besides these two reference systems considered by Einstein in 1905, also the obligatory choice of the planar or non-planar preferred P -ray. Such relative velocity we call, in what follows, the *ternary* velocity,

$$\mathbf{v} = \mathbf{v}(\text{privileged=preferred } P, \text{ initial } R, \text{ final } S). \quad (\text{IV.8})$$

If the unique solution of the link equation would be planar only (as it is for $\text{dim} = 2$, and the wishful-thinking for dimensions ≥ 3 [Ungar 2001]), therefore, we would be in heavens. Everything in special relativity would be crystal clear: unique boost imply unique (Einstein's reciprocal) relative velocity among reference systems R and S , and uniqueness of the relative

velocity must imply the associative addition of velocities. Whereas Ungar discovered in 1988 [p. 71] that addition of Einstein ‘unique velocities’ is *non*-associative.

IV.3 Noteworthy. Van Wyk in 1986, used implicitly the bivector-parametrization, and concludes that even a given *pair* of arbitrary initial vectors and a given pair of arbitrary final vectors do *not* provide enough information to specify unique Lorentz-link-boost. In spite of this, in [Ungar 2001, Theorem 11.16 on page 348], there is a suggestion (incorrect) that the Lorentz boost-link in four spacetime dimensions is uniquely given by single initial vector and single final vector.

V. ORTHOGONAL PRESENTATION

The P -link, from the given vector R to the given vector S , as given by Theorem III.3, has more transparent interpretation in the orthogonal presentation of the simple bivector as discussed in Subsection II A.

The planar-link (IV.4)-(IV.5), is given uniquely in terms of the initial vector R and the final vector S . The bivector (IV.3), generating the planar-isometry-link, has the following presentations

$$\frac{S \wedge R}{S^2} = \{(\text{id} - r)(S - R)\} \wedge \frac{R}{R^2} = \frac{S}{S^2} \wedge \{(\text{id} - s)(R - S)\}. \quad (\text{V.1})$$

If additionally $R \cdot S \neq 0$, then it is convenient for the future discussion to introduce the following vector that is non-skew-symmetric linear span of R and S , *i.e.* it is ‘non-Galilean subtraction’,

$$\varpi(R, S) \equiv \frac{R^2}{R \cdot S} (\text{id} - r)(S - R) \neq -\varpi(S, R), \quad R \cdot \varpi(R, S) = 0, \quad (\text{V.2})$$

$$\{\varpi(R, S)\}^2 = -R^2 + \frac{R^6}{(R \cdot S)^2}, \quad (R \cdot S)^2 = \frac{R^6}{R^2 + \{\varpi(R, S)\}^2}, \quad (\text{V.3})$$

$$\frac{1}{R^2} S \wedge R = \frac{1}{R^4} R \wedge (-R \cdot S) \varpi(R, S). \quad (\text{V.4})$$

Note that the scalar magnitude of the above binary vector (V.2)-(V.3), is symmetric relative to the exchange $R \leftrightarrow S$. The vector field (vector) (V.2)-(V.3) generalize the concept of the binary relative velocity, introduced in [Świerk 1988; Matolcsi 1994, §4.3; Bini 1995; Gottlieb 1996]. The expression (V.2) we will interpret later on in the four-dimensions that ‘the velocity $\varpi(R, S)$ of S relative R ’ is g -perpendicular to ‘4-vector-velocity R ’. With this

respect it is interesting to compare with what is emphasized by Cui in interesting paper [Cui 2006], that instead of the relative velocity, the force \simeq acceleration should be g -orthogonal to an observer R , as it is exemplified by the Lorentz force in electromagnetic field.

For arbitrary non-planar P -link $R \mapsto S$, (III.7)-(III.8)-(III.9), the simple bivector generating isometry-link, is manifestly reciprocal (exactly skew-symmetric) relative to the exchange $R \leftrightarrow S$,

$$\begin{aligned} (\mu P) \wedge (R - S) &= \mu P \wedge \{(\text{id} - p)(R - S)\} \\ &= \mu \left\{ \left(\text{id} - \frac{(R - S) \otimes g(R - S)}{(R - S)^2} \right) P \right\} \wedge (R - S), \end{aligned} \quad (\text{V.5})$$

$$\{(\text{id} - p)(R - S)\}^2 = \frac{1}{P^2} (P \wedge (R - S))^2. \quad (\text{V.6})$$

In terms of the ternary vector $W = W(P, R, S) \equiv \mu(\text{id} - p)(R - S)$, the arbitrary P -link $R \mapsto S$, has the explicit form given by (II.29), and these expressions (II.29), now must be supplemented by the following P -link identity

$$L_{P \wedge W} R = S, \quad W|_{P \wedge R \wedge S=0} = \frac{4(R + S)^2}{(R + S)^4 + 4(S \wedge R)^2} \frac{P}{P^2} \cdot (S \wedge R). \quad (\text{V.7})$$

We stress again that all results of entire all previous Sections holds for dimensions of the vector space (or module) ≥ 2 , and for arbitrary signature of the invertible symmetric metric tensor g (tensor field).

VI. THE EINSTEIN RECIPROCAL RELATIVE VELOCITY

In this Section we consider four-dimensional space-time with the Lorentzian signature of the metric tensor g , $(-+++)$. The main results of the previous Sections will be specified to this signature.

What could be a category of the reference systems in the Einstein and Minkowski relativity? An object of a category of reference systems can be either time-like monad, or, a tetrad=frame, $\tau\varepsilon\tau\rho\alpha\delta\sigma$, the Lorentz ortho-normal $(1+1+1+1)$ -split, the Lorentz frame=basis, as it is for example in [Misner, Thorne, Wheeler 1973], where, following Einstein, the coordinate system, and the reference system given by coordinate-free tetrad, is not always distinguished. Every morphism, and each change of a reference system, is the

Lorentz isometric transformation. The relative velocity among reference systems is defined in terms of the isometric Lorentz-boost, as a (not unique, P -seen) boost-linking two given time-like ‘four’-vectors, see (III.7).

VI.1 Noteworthy (Fahline 1982). Robertson and Noonan [1968, p. 66], and Fahline [1982], reexpressed the pure Lorentz coordinate transformation, *i.e.* the boost parameterized by ‘velocity of one reference frame relative to another’, as was presented by Einstein in 1905, in terms of the initial time-like reference vector R , and the final time-like reference vector S .

When the vectors R and S are both time-like, $R^2 = S^2 = -1$, and future-directed, then $R \cdot S \leq -1$. In this case in formula (II.17): $\gamma + 1 = 1 - R \cdot S$. Then, the isometry $L_{R \wedge S}$ (II.18)-(IV.4) coincide with the Fahline expression of pure Lorentz transformation (*i.e.* the boost), [Fahline 1982; Matolcsi 1993, 1.3.8; Matolcsi and Goher 2001 formula (27) on p. 91],

$$L_{R \wedge S} = \text{id} - 2 S \otimes gR + \frac{(R + S) \otimes g(R + S)}{1 - R \cdot S}. \quad (\text{VI.1})$$

The above expression for g -isometry (VI.1), Fahline derived from the Lorentz coordinate transformation.

The Lorentz boost is known variously but synonymously as: rotation on imaginary angle [Sommerfeld 1909], non-rotational Lorentz transformation [Fisher 1972], pure Lorentz transformation [Fahline 1982]. In what follows we will restrict the name Lorentz boost for isometry-link parameterized explicitly in terms of the *relative velocity*. For this we need to define what means a ‘relative velocity’ in connection with basis-free isometry-link?

VI.2 Definition (Isometric velocity). The relative velocity parameterizing the isometry, $L(\mathbf{v}) \in O_g$, $L(\mathbf{v} = \mathbf{0}) \equiv \text{id}$, is said to be *isometric-velocity*, or, the Einstein velocity. The isometric-velocity must be reciprocal $\{L(\mathbf{v})\}^{-1} = L(-\mathbf{v})$,

$$L(\mathbf{v})R = S \quad \iff \quad L(-\mathbf{v})S = R. \quad (\text{VI.2})$$

Reader interested in special relativity of Albert Einstein [Einstein 1905], have the right to ask, why bivector parametrization, (II.12)-(II.18)-(II.29)-(III.7), is important? Following Lorentz [1904], Poincaré [1904], and Einstein [1905], the textbooks parameterizes Lorentz boost in terms of a single vector ‘of relative velocity’ that it is suggested to be outside of the

Minkowski spacetime, in some idealized absolute Euclidean space. This assume that exists 'standard' embedding, given by heavens, $O(3) \hookrightarrow O(1, 3)$.

However, all observed relative velocities are tangent to simultaneity of observer, assuming that an observer can not see moved bodies in his own past or in his own future. Therefore relative velocities *are* space-like vectors *in* Minkowski space-time. A single vector can parameterize an isometry if an isometry L is a reflection = unipotent (II.7)-(III.4), or, if we chose the preferred embedding $O(3) \hookrightarrow O(1, 3)$. Otherwise, we need at least a simple bivector, that is relevant for parametrization of non-unipotent isometry. Many Authors consider the matrix representation of the Lorentz boost as 'the best definition', for example in [Ungar 1988 page 62; 2001, page 254]. In fact it is just contrary, the matrix representation, irrelevant-basis-dependent, introduce irrelevant information, whereas the relevant bivector is lost or it is hidden, and therefore matrix representation does not elucidate the correct meaning of the Einstein 'relative velocity' parameterizing Lorentz isometry. Our aim is to show the place of this simple bivector in Lorentz-boost transformations.

Bivectors are relevant for *every* isometry. For the *Lorentz* isometry group, the relevance of the bivectors was emphasized from another perspective by Ehlers, Rindler and Robinson [1966], [Rindler and Robinson 1999], and by Baylis and Sobczyk [2004].

A massive observer, a reference system, we identify with time-like future-directed and normalized vector fields, $P^2 = -1$. The simultaneity of an observer P is given by g -dependent differential Pfaffian one-form $-gP$. All observers by definition are g -orthogonal in this meaning, all observers are orthogonal projectors, *i.e.* are orthogonal idempotents.

For Lorentzian signature, if a vector P is time-like, and $P \cdot V = 0$, then vector V is space-like.

VI.3 Definition (Observing velocity). For a time like vector P (representing massive reference system), and a space-like vector \mathbf{v} (representing the possible relative velocity), the condition $P \cdot \mathbf{v} = 0$, is interpreted as the necessary and sufficient for observing \mathbf{v} by P .

The orthogonality condition $P \cdot \mathbf{v} = 0$, means that the vector \mathbf{v} is tangent to simultaneity of time-like observer, $P^2 = -1$. We assume that every observer see the relative velocities among bodies, as tangent to his simultaneity, and not in his past or in his future.

The Heaviside-FitzGerald-Lorentz scalar factor is denoted by

$$\gamma_{\mathbf{v}} \equiv \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-\frac{1}{2}}, \quad \frac{\mathbf{v}^2}{c^2} = \frac{\gamma_{\mathbf{v}}^2 - 1}{\gamma_{\mathbf{v}}^2}, \quad \bar{\mathbf{v}} \equiv \gamma_{\mathbf{v}} \frac{\mathbf{v}}{c}, \quad \bar{\mathbf{v}}^2 = \gamma_{\mathbf{v}}^2 - 1. \quad (\text{VI.3})$$

Let's come back to bivector (V.5), solving the isometry-link subject to Condition III.2,

$$L_{(\mu P) \wedge (R-S)} R = S, \quad (\mu P) \wedge (R - S) = P \wedge \mu(\text{id} - p)(R - S). \quad (\text{VI.4})$$

VI.4 Definition (Ternary isometric relative velocity). Let the system of three time-like vectors (vector fields) $\{P, R, S\}$ represents the three-body massive system, *i.e.*

$$P^2 = R^2 = S^2 = -1.$$

The ternary isometric velocity $\mathbf{v} = \mathbf{v}(P, R, S)$ of a massive body S relative to massive body R , as observed by a massive reference system P , is defined as follows

$$\bar{\mathbf{v}} \equiv \gamma_{\mathbf{v}} \frac{\mathbf{v}}{c} \equiv \mu(\text{id} - p)(R - S)|_{P^2=R^2=S^2=-1}, \quad (\text{VI.5})$$

$$L_{P \wedge \bar{\mathbf{v}}} R = S \quad \implies \quad \text{unique} \quad \bar{\mathbf{v}} = \mu(\text{id} - p)(R - S). \quad (\text{VI.6})$$

Lemma VI.1. *The following formulas holds*

$$\gamma_{P \wedge \bar{\mathbf{v}}} = \gamma_{\mathbf{v}}, \quad \gamma_{(\mu P) \wedge (R-S)} = \left| \frac{\{P \cdot (R + S)\}^2 - \{P \wedge (R - S)\}^2}{\{P \cdot (R + S)\}^2 + \{P \wedge (R - S)\}^2} \right|, \quad (\text{VI.7})$$

$$\gamma_{(\mu P) \wedge (R-S)}|_{P \wedge R \wedge S=0} = |R \cdot S|, \quad (\text{VI.8})$$

$$\bar{\mathbf{v}}|_{P \wedge R \wedge S=0} = \frac{4(R + S)^2}{(R + S)^4 + 4(S \wedge R)^2} \frac{P}{P^2} \cdot (S \wedge R) = P \cdot (S \wedge R). \quad (\text{VI.9})$$

The isometric velocity $\mathbf{v} = \mathbf{v}(P, R, S)$ (VI.6) is P -dependent velocity of S relative to the reference system R .

All vector fields (vectors) of the system $\{P, R, S\}$ must be time-like normalized in order that the bivector, $(\mu P) \wedge (R - S)$, generating an isometry $R \mapsto S$, can be interpreted in terms of the physical relative velocity $\mathbf{v} = \mathbf{v}(P, R, S)$ among massive bodies. Note that no one time-like vector field in the ternary system $\{P, R, S\}$ needs to be inertial.

Is evident that the Einstein relative isometric planar or no-planar relative ternary velocity, $\mathbf{v}(P, R, S)$, is not given uniquely by the reference systems R and S , because needs the choice of the preferred observer P in Definition (VI.6) and in Theorem (VII.1). The Einstein velocity depends on the choice of the *preferred* massive observed P . One can say that the

isometric and reciprocal velocity of S relative to R is *seen* by preferred observer P . This is why the relative isometric-velocity (VI.5) is said to be the *ternary velocity*. This is a velocity of a reference system S relative to R as *seen* by P . For the fixed reference systems, R and S , and the variable choice of the preferred observer P , there could be infinite many Einstein's velocities of S relative to R , as *seen* by many different privileged variable observers P .

VI.5 Definition (Lorentz boost). Let $P^2 = -1$ and $P \cdot \mathbf{v} = 0$. Let introduce the following (differential) forms,

$$\nu \equiv (\gamma_{\mathbf{v}} - 1)gP - g\bar{\mathbf{v}}, \quad \xi \equiv gP - \frac{g\bar{\mathbf{v}}}{\gamma_{\mathbf{v}} + 1}. \quad (\text{VI.10})$$

A basis-free boost is a pure Lorentz isometry transformation parameterized in terms of the bounded space-like velocity \mathbf{v} in terms of the bivector $P \wedge \gamma_{\mathbf{v}}\mathbf{v}/c$, as follows

$$\begin{aligned} L_{P \wedge \bar{\mathbf{v}}} &= \text{id} - P \otimes \nu - \bar{\mathbf{v}} \otimes \xi \\ &= \text{id} - (\gamma_{\mathbf{v}} - 1)P \otimes gP + \gamma_{\mathbf{v}} \left(P \otimes g\frac{\mathbf{v}}{c} - \frac{\mathbf{v}}{c} \otimes gP \right) + \frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}} + 1} \frac{\mathbf{v} \otimes g\mathbf{v}}{c^2}, \end{aligned} \quad (\text{VI.11})$$

For the given space-like bounded velocity \mathbf{v} , there is a two-dimensional manifold of allowed time-like massive \mathbf{v} -observers $\{P\}$ in (VI.11) (observer of \mathbf{v} is *not* unique), because in dimension four the set of two conditions, $P \cdot \mathbf{v} = 0$ and $P^2 = -1$, has the two-parameter family of solutions. The observer-*dependent* Lorentz boost (VI.11) must be compared with the frequent definition of the Lorentz boost as the matrix parameterized 'uniquely' in terms of the velocity \mathbf{v} only, see for example in [Møller 1952, 1972; Ungar 1988, page 62, formulas (6)-(8); Ungar 2001, page 254, formulas (8.2)-(10.29)].

From the boost definition (VI.11), the following expressions follows,

$$\begin{aligned} L_{P \wedge \bar{\mathbf{v}}} R &= R - \left\{ (\gamma_{\mathbf{v}} - 1)(P \cdot R) - \gamma_{\mathbf{v}} \left(\frac{\mathbf{v}}{c} \cdot R \right) \right\} P - \\ &\quad \gamma_{\mathbf{v}} \left\{ (P \cdot R) - \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}} + 1} \left(\frac{\mathbf{v}}{c} \cdot R \right) \right\} \frac{\mathbf{v}}{c}, \end{aligned} \quad (\text{VI.12})$$

$$\{L_{P \wedge \bar{\mathbf{v}}}\}^{-1} = L_{P \wedge (-\bar{\mathbf{v}})}, \quad P \wedge R \wedge (L_{P \wedge \bar{\mathbf{v}}} R) = -(\xi R)P \wedge R \wedge \bar{\mathbf{v}}, \quad (\text{VI.13})$$

$$R \cdot (L_{P \wedge \bar{\mathbf{v}}} R) = R^2 - (\gamma_{\mathbf{v}} - 1)(P \cdot R)^2 + \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}} + 1} \left(\frac{\mathbf{v}}{c} \cdot R \right)^2, \quad (\text{VI.14})$$

$$L_{P \wedge \bar{\mathbf{v}}} P = \gamma_{\mathbf{v}} \left(P + \frac{\mathbf{v}}{c} \right). \quad (\text{VI.15})$$

VII. LORENTZ COORDINATE TRANSFORMATIONS

In what follows we assume that R is time-like vector field such that $R^2 = -1$ (a massive observer), with an associated idempotent operator, $r \equiv R \otimes (-gR)$. Then $L_{P \wedge \bar{\mathbf{v}}} R$ is a massive body that moves with the velocity \mathbf{v} relative to R , as measured by reference system P . Let these two observers, R and LR , are observing the particle=event e . Then with respect to the reference system r , particle e is

$$e = re + (\text{id} - r)e = ctR + \mathbf{x} = ct'LR + \mathbf{x}', \quad (\text{VII.1})$$

$$ct \equiv -R \cdot e \quad \text{and} \quad ct' \equiv -(LR) \cdot e = -R \cdot (L^{-1}e). \quad (\text{VII.2})$$

Here, $R \mapsto LR$, is isometry action that change the reference system, and, $e \mapsto L^{-1}e$, is the change of the observed coordinated particle.

The massive bodies R and $S \equiv LR$, understood as the normalized time-like vector fields, can be identified with the pair of the reference systems, in mutual motion, considered by Albert Einstein in 1905.

Then $\mathbf{x} = \mathbf{x}(r, e) \equiv (\text{id} - r)e$ is directly observable coordinate=location of a particle=event e relative to the reference system R . Correspondingly, $\mathbf{x}' = \mathbf{x}(S = LR, e)$ be a coordinate of the same particle=event e as seen by the reference system $LR = S$.

We wish to apply the basis-free boost (VI.11), for the textbooks's case

$$L_{P \wedge (-\bar{\mathbf{v}})} \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix} = L_{P \wedge (-\bar{\mathbf{v}})}(ctR + \mathbf{x}) = \begin{pmatrix} ct' \\ \mathbf{x}' \end{pmatrix} = ct'R + \mathbf{x}'. \quad (\text{VII.3})$$

How the choice of the preferred time-like observer P , observing the velocity \mathbf{v} , $P \cdot \mathbf{v} = 0$, would appears in Lorentz transformation of coordinates of particle e observed by R and LR ?

VII.1 Theorem (Lorentz coordinate transformation). *The general isometric pure Lorentz transformation of coordinates of an event e as seen by the reference systems R and LR respectively, is parameterized by bivector $P \wedge \bar{\mathbf{v}}$, [and not by 'a velocity parameter \mathbf{v} '*

alone], and depends on preferred observer P ,

$$\nu e = ct\{(\gamma_{\mathbf{v}} - 1)P \cdot R + \bar{\mathbf{v}} \cdot R\} + \bar{\mathbf{v}} \cdot \mathbf{x} + (\gamma_{\mathbf{v}} - 1)P \cdot \mathbf{x}, \quad (\text{VII.4})$$

$$\xi e = ct \left(P \cdot R + \frac{\bar{\mathbf{v}} \cdot R}{\gamma_{\mathbf{v}} + 1} \right) + \frac{\bar{\mathbf{v}} \cdot \mathbf{x}}{\gamma_{\mathbf{v}} + 1} + P \cdot \mathbf{x}, \quad (\text{VII.5})$$

$$ct' = ct + (gR)\{(\nu e)P - (\xi e)\bar{\mathbf{v}}\}, \quad (\text{VII.6})$$

$$\mathbf{x}' = \mathbf{x} - (\text{id} - r)\{(\nu e)P - (\xi e)\bar{\mathbf{v}}\}, \quad (\text{VII.7})$$

$$(\text{id} - L^{-1})e = (\nu e)P - (\xi e)\bar{\mathbf{v}}. \quad (\text{VII.8})$$

Thus the above Lorentz transformation, when compared with the traditional presentations [Einstein 1905; Fock 1955, 1959, 1961, 1964 §10; Jackson 1962, 1975 §11.3, Zachary and Gill, the present volume], depends on three new scalars:

- $R \cdot P$ - describe the mutual motion among reference system R and preferred observer P .
- $R \cdot \mathbf{v}$ - describe how much the relative velocity observed by P is far away from simultaneity of the reference system given by time-like normalized vector field R .
- $P \cdot \mathbf{x}$ - describe how much the position of an event e relative to the reference system R , is far away from simultaneity of the preferred observer P .

In the Gallilean limit of the absolute simultaneity, $c^2 \rightarrow \infty$, $R \cdot \mathbf{v} = 0$, Theorem (VII.1) gives $t' = t$, and $\mathbf{x}' = \mathbf{x} - t\mathbf{v}$.

The 'singular' case of the coplanar preferred observer, and the general case of non-planar preferred observer, deserve to be consider separately, (VI.13),

$$\begin{aligned} P \wedge R \wedge (L_{P \wedge \bar{\mathbf{v}}} R) &= -(\xi R)P \wedge R \wedge \bar{\mathbf{v}} \\ &= \begin{cases} 0 & \text{for coplanar preferred observer } P \\ \neq 0 & \text{for non-planar preferred observer } P. \end{cases} \end{aligned} \quad (\text{VII.9})$$

Moreover, for non-planar preferred P , still the particular case when P and R are in mutual motion $P \cdot R \neq -1$, however both can observe the *same* velocity $R \cdot \mathbf{v} = 0 = P \cdot \mathbf{v}$, also deserve the separate consideration. We will consider the entire analysis of general non-planar Lorentz transformation in a separate publication.

Let a preferred observer P be planar relative to the plane $R \wedge LR$,

$$P \wedge R \wedge (L_{P \wedge \bar{\mathbf{v}}} R) = -(\xi R) P \wedge R \wedge \bar{\mathbf{v}} = 0. \quad (\text{VII.10})$$

If $\xi R \neq 0$ and $(R \cdot \bar{\mathbf{v}})^2 + \gamma_{\mathbf{v}} - 1 \neq 0$, then the Lorentz transformation is parameterized by two additional scalar parameters, $R \cdot P$ and $R \cdot \mathbf{v}$, only, because there is the following identity,

$$\{(R \cdot \bar{\mathbf{v}})^2 + \gamma_{\mathbf{v}}^2 - 1\} P \cdot \mathbf{x} = (P \cdot R)(R \cdot \bar{\mathbf{v}})(\mathbf{x} \cdot \bar{\mathbf{v}}). \quad (\text{VII.11})$$

VII.2 Example (Einstein 1905; Fock 1955, 1959, 1961, 1964 §10; Jackson 1962, 1975 §11.3). Let $P = R$. Then all three additional scalars are fixed: $(\text{id} - r)P = 0$, $P \cdot R = -1$, $R \cdot \mathbf{v} = 0$, and $P \cdot \mathbf{x} = 0$. Therefore \mathbf{v} is a velocity of a reference system $S = LR$ relative to reference system R as *seen* by $P = R$. The Lorentz-boost isometry-transformation of coordinates of an event e , is as follows

$$\mathbf{x}' = \mathbf{x} + \frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}} + 1} \frac{(\mathbf{v} \cdot \mathbf{x})}{c^2} \mathbf{v} - \gamma_{\mathbf{v}} \mathbf{v} t, \quad t' = \gamma_{\mathbf{v}} \left(t - \frac{\mathbf{v} \cdot \mathbf{x}}{c^2} \right), \quad (\text{VII.12})$$

$$\frac{\mathbf{v} \cdot \mathbf{x}}{c^2} = t - \frac{t'}{\gamma_{\mathbf{v}}}, \quad \mathbf{v} = \left(\frac{\gamma_{\mathbf{v}} + 1}{\gamma_{\mathbf{v}}} \right) \left(\frac{\mathbf{x} - \mathbf{x}'}{t + t'} \right), \quad \gamma_{\mathbf{v}} = \frac{1 + \frac{1}{c^2} \left(\frac{\mathbf{x} - \mathbf{x}'}{t + t'} \right)^2}{1 - \frac{1}{c^2} \left(\frac{\mathbf{x} - \mathbf{x}'}{t + t'} \right)^2}, \quad (\text{VII.13})$$

$$\mathbf{v} = \mathbf{v}(R, R, LR) = 2 \left\{ 1 + \frac{1}{c^2} \left(\frac{\mathbf{x} - \mathbf{x}'}{t + t'} \right)^2 \right\}^{-1} \left(\frac{\mathbf{x} - \mathbf{x}'}{t + t'} \right), \quad (\text{VII.14})$$

$$\mathbf{v} = \mathbf{v}(R, R, LR) = -\mathbf{v}(R, LR, R). \quad (\text{VII.15})$$

In the Galilean limit of the absolute simultaneity, $c^2 \rightarrow \infty$, $t' = t$ and the Urbantke formula (VII.14) gives $\mathbf{v} = \frac{\mathbf{x} - \mathbf{x}'}{t}$.

VIII. NON-ASSOCIATIVE ADDITION OF ISOMETRIC RELATIVE VELOCITIES

The explicit coordinate expression (VII.14) for the relative velocity parameterizing the Lorentz coordinate transformation, was derived, in another way, by Urbantke [2003, p. 115, formula (7)], and in another form using gyration by Ungar [2001, p. 348, Theorem 11.16]. The Urbantke expression can be interpreted as the operational *definition* of the Einstein relative velocity in terms of the directly measurable space-distances and time-intervals for the case when the preferred observer (Earth) is chosen to be $S = R$, compare with the general definition of the ternary relative velocity (VI.5). The Urbantke expression (VII.14)

shows that the relative velocity, parameterizing the planar isometry transformation, must be reciprocal, $\mathbf{v}^{-1} = -\mathbf{v}$. This relative isometric-velocity is *ternary* velocity because *assume* that preferred observer is $P = R$. For the Lorentz group property, $(L(\mathbf{v}))^{-1} = L(-\mathbf{v})$, the reciprocity of the velocity is unavoidable, and in fact imply that the link-transformation $R \mapsto S$ must be an isometry. In particular, one can choose in Definition (VI.5) for example $P = S = LR$, *i.e.* consider the reference system S to be preferred, however the reciprocity can not be lost,

$$\mathbf{v}(S, R, S) = -\mathbf{v}(S, S, R) \neq \mathbf{v}(R, S, R). \quad (\text{VIII.1})$$

How to interpret the Urbantke formula (VII.13)-(VII.14)?

The relative velocity \mathbf{v} among reference systems for the choice of the preferred *planar* observer $P + R$, (VII.12)-(VII.14), does not involve the distance among R and system S . Instead, this velocity of S relative to R , needs the two auxiliary distances, $\mathbf{x}(R, e)$ and $\mathbf{x}(S, e)$, among event e and the both reference systems in question. Therefore one can expect to interpret this reciprocal velocity \mathbf{v} of S relative to R as seen by R , as the kind of the ‘subtraction’ from the actually measured pair of velocities, $\mathbf{u} \equiv \mathbf{u}(R, e)$ and $\mathbf{w} \equiv \mathbf{u}(S, e)$, of the event e relative to the reference systems R and S , correspondingly.

The extra dependence of the Einstein relative velocities on preferred observer, (VI.6), implies that the addition of the relative isometric-velocities (Lorentz links among monads or links among Lorentz frames = bases), is non-associative. The non-associative addition of Einstein’s isometric velocities [Einstein 1905, Fock 1955], following Vargas [1984, p. 647] and Mocanu [1986, 1992], is denoted by a symbol \oplus .

The appropriate subtraction of relative velocities, giving \mathbf{v} in terms of \mathbf{u} and \mathbf{w} , is equivalent, implicitly, to the addition of velocities, expressing for example $\mathbf{w} = \mathbf{u} \oplus (-\mathbf{v})$, as the addition of $-\mathbf{v}$ with \mathbf{u} . We will use the following identity

$$(\mathbf{v} \cdot \mathbf{u})\mathbf{v} = \mathbf{v} \cdot (\mathbf{u} \wedge \mathbf{v}) + c^2 \left(1 - \frac{1}{\gamma_{\mathbf{v}}^2}\right) \mathbf{u}. \quad (\text{VIII.2})$$

The addition of relative reciprocal isometric-velocities was derived by Einstein in 1905 using composition of the Lorentz coordinate transformations (assuming that composed velocities are constant). The same addition formula was derived by Vladimir Fock using the Lorentz transformation of the differentials of coordinates (VII.12), the first differential prolongation of (VII.12), and assuming the constant isometric relative velocity, $\mathbf{v} = \text{const}$,

$d\mathbf{v} = 0$, only. The equivalent way to derive the addition \oplus of relative velocities is by means of the differential prolongation of the formula (VII.13) (instead of (VII.12)),

$$d\mathbf{v} = 0 \iff d\left(\frac{\mathbf{x} - \mathbf{x}'}{t + t'}\right) = 0, \quad d\mathbf{x} = \mathbf{u} dt \quad \text{and} \quad d\mathbf{x}' = \mathbf{w} dt', \quad (\text{VIII.3})$$

$$\mathbf{w} = \mathbf{u} \oplus (-\mathbf{v}) = \frac{\mathbf{u} - \gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{v}}\left(1 - \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}\right)} + \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}} + 1} \frac{(\mathbf{v}\cdot\mathbf{u})\mathbf{v}}{(c^2 - \mathbf{v}\cdot\mathbf{u})}, \quad (\text{VIII.4})$$

$$= \frac{\mathbf{u} - \mathbf{v}}{\left(1 - \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}\right)} + \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}} + 1} \frac{\mathbf{v}\cdot(\mathbf{u} \wedge \mathbf{v})}{(c^2 - \mathbf{v}\cdot\mathbf{u})}, \quad \mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v} = 0. \quad (\text{VIII.5})$$

The Einstein-Fock addition of velocities (VIII.4) is an implicit expression of the Lorentz parametria \mathbf{v} in terms of relative velocities \mathbf{u} and \mathbf{w} . The explicit subtraction one can derive using the following naive algebraic definition of relative velocities \mathbf{u} and \mathbf{w} , without differentials, like e.g. in [Ungar 2001, Theorem 11.16 on page 348],

$$\mathbf{x} = \mathbf{u} t \quad \text{and} \quad \mathbf{x}' = \mathbf{w} t'. \quad (\text{VIII.6})$$

Inserting these definitions into interval, $e^2 = (Le)^2$, $-(ct')^2 + \mathbf{x}'^2 = -(ct)^2 + \mathbf{x}^2$, we get the reciprocal velocity \mathbf{v} as the subtraction of \mathbf{w} from \mathbf{u} ,

$$(t' \gamma_{\mathbf{u}})^2 = (t \gamma_{\mathbf{w}})^2, \quad \gamma_{\mathbf{v}} \equiv \frac{(\gamma_{\mathbf{u}} + \gamma_{\mathbf{w}})^2 + \frac{1}{c^2}(\gamma_{\mathbf{u}}\mathbf{u} - \gamma_{\mathbf{w}}\mathbf{w})^2}{(\gamma_{\mathbf{u}} + \gamma_{\mathbf{w}})^2 - \frac{1}{c^2}(\gamma_{\mathbf{u}}\mathbf{u} - \gamma_{\mathbf{w}}\mathbf{w})^2}, \quad (\text{VIII.7})$$

$$\frac{\gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{v}} + 1} \equiv \frac{\gamma_{\mathbf{u}}\mathbf{u} - \gamma_{\mathbf{w}}\mathbf{w}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{w}}}. \quad (\text{VIII.8})$$

The above expression one can compare with the Ungar Theorem 11.16, where it is assumed implicitly that preferred observer is coplanar $P = R$.

We need to stress that all three relative velocities, \mathbf{v} , \mathbf{u} , and \mathbf{w} , entering the addition law (VIII.4)-(VIII.5), and entering the subtraction operation, are reciprocal, and are ternary velocities as *seen* by P , $P \cdot \mathbf{v} = P \cdot \mathbf{u} = P \cdot \mathbf{w} = 0$, all these velocities are tangent to simultaneity of P . Elsewhere we derive the same addition law of the reciprocal velocities (VIII.4)-(VIII.5), without assuming that the relative velocity \mathbf{v} is constant.

A. Second differential prolongation

The second differential prolongation of the addition of velocities (VIII.4) gives

$$d\mathbf{u} = \mathbf{a} dt \quad \text{and} \quad d\mathbf{w} = \mathbf{a}' dt', \quad (\text{VIII.9})$$

$$\gamma_{\mathbf{v}}^2 \left(1 - \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}\right)^2 \mathbf{a}' = \mathbf{a} + \frac{\mathbf{v}\cdot\mathbf{a}}{c^2 - \mathbf{v}\cdot\mathbf{u}} \left(\mathbf{u} - \frac{\gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{v}} + 1}\right). \quad (\text{VIII.10})$$

There are the following two identities

$$\frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}} + 1} \{(\mathbf{v} \cdot \mathbf{a})\mathbf{v} - \mathbf{v} \cdot (\mathbf{a} \wedge \mathbf{v})\} = \left(1 - \frac{1}{\gamma_{\mathbf{v}}}\right) \mathbf{a}, \quad (\text{VIII.11})$$

$$\frac{\gamma_{\mathbf{v}}^2 - 1}{\gamma_{\mathbf{v}}^2} \{(\mathbf{v} \cdot \mathbf{a})\mathbf{u} - (\mathbf{v} \cdot \mathbf{u})\mathbf{a}\} = (\mathbf{v} \cdot \mathbf{u})\{\mathbf{v} \cdot (\mathbf{a} \wedge \mathbf{v})\} - (\mathbf{v} \cdot \mathbf{a})\{\mathbf{v} \cdot (\mathbf{u} \wedge \mathbf{v})\}. \quad (\text{VIII.12})$$

Torres del Castillo and Pérez Sánchez [2006] consider a particle e accelerated relative to reference systems R and S . Authors consider the collinear motion only, $\mathbf{v} \wedge \mathbf{u} = 0$, and then (VIII.10) gives,

$$\gamma_{\mathbf{v}}^3 \left(1 - \frac{\mathbf{v} \cdot \mathbf{u}}{c^2}\right)^3 \mathbf{a}' = \mathbf{a}. \quad (\text{VIII.13})$$

IX. THE EINSTEIN PRINCIPLE OF RECIPROCITY OF RELATIVE VELOCITY

Throughout this paper, the Galilean binary addition of velocities is denoted by a binary operation symbol ‘+’.

The Einstein formulation of special relativity needs ‘the Einstein principle of reciprocity of velocity’. This principle is related to the identification of the Lorentz-group *parametrization* of the frames/Koordinatensystem’s with the physical inertial observers. This identification motivates Einstein’s reciprocity-axiom [1905, §3]:

$$\mathbf{v} \oplus \mathbf{u} = \mathbf{0} \quad \iff \quad \mathbf{v} + \mathbf{u} = \mathbf{0} \quad \implies \quad \mathbf{v} \wedge \mathbf{u} = 0. \quad (\text{IX.1})$$

In words: ‘if the velocity of a frame S relative to R is \mathbf{u} , then the velocity of R relative to S is $-\mathbf{u}$.’

In the Einstein reciprocity axiom (IX.1) the zero velocity is the velocity of R with respect to R , $\mathbf{0} \equiv \mathbf{0}_R$. Terletsii [1966, 1968] and Mocanu [1986, 1992] refer to the Einstein axiom (IX.1) as ‘the Einstein principle of reciprocity of velocity’.

The Einstein principle of reciprocity is discussed by Cattaneo [1958], Berzi & Gorini [1968] and by Newburgh [1972].

The proper-time an exact differential form of an observer is given by an equivalence relation on events, two events are in the same equivalence class if they are simultaneous. The proper-time is relative, is observer-dependent. The Einstein principle (IX.1), that the inverse with respect to the addition of velocities \oplus must be the same as for the Galilean absolute time addition ‘+’, apparently contradicts the physics of the relativity of simultaneity. Mutually moved massive bodies/observers/systems must possess different simultaneous relations.

Therefore the relative velocities measured (in the time-space) by mutually moved observers must be tangent to the different not co-planar simultaneity foliations in a time-space.

The relative velocity that obey the Einstein reciprocity must be preferred observer-dependent, because such relative velocity must be tangent to simultaneity of the preferred observer, and not necessarily tangent to simultaneity of observed bodies in mutual motion.

IX.1 Noteworthy (Speed of the light). The limited velocity of the speed of light is believed to be the necessary axiom for the special relativity theory, cf. explicit second axiom in *Prinzip der Relativität*, stated by Albert Einstein on the first page of his fundamental paper [Einstein 1905]: speed of the light, \mathbf{c} , is independent on a speed \mathbf{v} of the radiating massive source,

$$\forall \mathbf{v}, \quad (\mathbf{c} \oplus \mathbf{v})^2 = \mathbf{c}^2 = (\mathbf{v} \oplus \mathbf{c})^2. \quad (\text{IX.2})$$

In fact, (IX.2) is the *consequence* of the law of addition, and the limiting velocity one can deduce as the theorem from the Einstein-Fock relativistic law of the addition of relative isometric velocities (VIII.4).

We do not agree with the following Adolf Grünbaum opinion,

This claim concerning the limiting character of the speed of light is a fundamental axiom of the theory and is *not* a theorem depending for its deduction on the relativistic law for the addition of velocities, as we are sometimes told. For (i) this addition law itself depends for its own deduction on the Lorentz transformations ...

Adolf Grünbaum in *Logical and Philosophical Foundations of the Special Theory of Relativity* [1955, 1960 p. 405].

To get the addition \oplus of relative velocities the Einstein two explicit axioms (from the first page of the Einstein fundamental paper) are either not sufficient, nor they are needed at all. We claim that the addition \oplus of Einstein's isometric velocities can not be deduced without the reciprocity axiom stated by Einstein almost at the end of §3 of his fundamental paper [Einstein 1905, §3].

In the relativity theory as the groupoid category, in the Lorentz-group-free relativity [Oziewicz 2006, Page 2006], there is a Theorem that the speed of the light is emitter-free for arbitrary *non-inertial* emitter.

X. LORENTZ-GROUP-FREE RELATIVITY

In this essay, relativity means relativity of time and relativity of space, and not yet the gravity theory. In most textbooks the special relativity means inertial observers and constant relative velocities, and assume matter-free flat Lorentzian metric tensor field with no curvature and no torsion, and with Poincaré symmetry group,

$$\text{No matter} \iff \text{no curvature} \iff \left\{ \begin{array}{c} \text{Empty spacetime} \\ \text{with} \\ \text{Poincaré symmetry} \end{array} \right\} \quad (\text{X.1})$$

In many respects our presentation follows the ideas of the monograph by Tomás Matolcsi [1994], where coordinate-free (= without reference frames) relativity theory of spacetime is formulated in the attractive way that does not need the artificial separation into special and general relativities [Matolcsi 1994, §6]. Similarly to [Matolcsi 1994] we consider most general non-inertial observers. Matolcsi define an observer as the normalized time-like vector field, monad, whereas we identify an observer with trace-class idempotent, like (IV.2).

A. Relative velocity as morphism in a connected groupoid category of null objects

A groupoid is a small category in which all morphisms are isomorphisms.

The Lorentz-group-free relativity theory holds for arbitrary metric tensor field and for the most general field of observers, not inertial, rotated, accelerated, deformed, etc. The Lorentz-group-free relativity theory, does not needs the concept of the Lorentz isometric transformations as the relativity transformations, however does needs some very elementary concepts of the category theory. The Lorentz-group-free relativity is a connected groupoid category of massive observers (a connected groupoid category of massive reference systems), and is denoted by ϖ .

An object of category ϖ , an observer, is (1+3)-split-idempotent like (IV.2) (in particular, a monad = $\chi\rho\nu\sigma\zeta$). A family of objects=observers generate an associative operator algebra $\text{Obs}(\varpi)$, that most likely is a Frobenius algebra. Each morphism in this category ϖ , a change of observer, is given by relative velocity among reference systems. This is non isometry transformation, so it is not Lorentz transformation, because the domain is restricted to the

sub-manifold of time-like normalized vector-fields only. The addition of velocities-morphisms is associative [Oziewicz 2005, 2006, Page 2006].

TABLE I: The different categories of reference systems.

	Special relativity	Lorentz-group-free relativity: groupoid category
object	monad or Lorentz basis $\tau\epsilon\tau\rho\alpha\sigma$	(1+3)-split split-idempotent $\chi\rho\nu\sigma\zeta$
morphism: change of observer	isometry: Lorentz coordinate transformation	relative velocity, not isometry
addition of relative velocities	non -associative	associative

Lorentz-group-free relativity of space and of proper-time does not need the concepts of isometry and the Lorentz group, nor the hyperbolic geometry of Lobachevsky [1829] and János Bolyai [1832], whereas needs some basic concepts of the category theory [Mac Lane 1998], and need the category of observers, as in [Vladimirov 1982].

The fundamental, primary concept, is the relative non-reciprocal velocity-morphism between massive split-idempotents.

$$\text{No matter} \iff \text{no relative velocity} \iff \text{no relativity} \quad (\text{X.2})$$

The Lorentz group, the symmetry group of the metric tensor for empty space-time, does not need to be the foundation of the conceptual relativity theory. Instead, the concept of non-inertial observer (reference system) seems to be more relevant.

A starting concept is an enriched connected groupoid category of (1+3)-observers/bodies, denoted by ϖ , with the categorical morphism interpreted as an actual relative velocity, this is the main axiom. The set of all arrows of a category ϖ with a given source $p \in \text{obj } \varpi$, and

a given target, $q \in \text{obj } \varpi$, is abbreviated in the following different ways:

$$\varpi(p, q) = \text{hom}_{\varpi}(p, q) = \text{hom}(p, q) = \varpi(\text{observer}, \text{observed}). \quad (\text{X.3})$$

We are using the first notation. The composition of morphisms reads from right to left,

$$\varpi(q, r) \times \varpi(p, q) \xrightarrow{\circ} \varpi(p, r), \quad (\text{X.4})$$

$$p \xrightarrow{\varpi(p, q)} q \xrightarrow{\varpi(q, r)} r. \quad (\text{X.5})$$

The addition of the velocities-morphisms (not parameters in the Lorentz boosts) is the associative operation and is denoted by \circ .

If a velocity \mathbf{u} is a velocity of a body/object $q \in \text{obj } \varpi$ with respect to a laboratory/observer $p \in \text{obj } \varpi$, we display (picturing) this velocity \mathbf{u} as an actual categorical arrow/morphism/directed-path starting at laboratory/observer/source p (as a node of the directed graph), and ending at a body/target q ,

$$p \xrightarrow{\mathbf{u}} q \quad \text{or} \quad \mathbf{u} = \varpi(p, q), \quad (\text{X.6})$$

$$\text{observer/source/tail/domain of } \mathbf{u} = p, \quad (\text{X.7})$$

$$\text{observed/target/head/codomain } \mathbf{u} = q, \quad (\text{X.8})$$

$$\mathbf{u} = \mathbf{0} \iff p \equiv q, \quad \mathbf{0}_p \equiv \varpi(p, p). \quad (\text{X.9})$$

This reads: the velocity \mathbf{u} is the (unique!) relative velocity of a massive body q as measured/seen/observed by a massive laboratory/body p . A category symbol ϖ is interpreted as the physical process of a measurement of a velocity. Two bodies in relative rigid rest are considered as one object/body of groupoid category ϖ .

For any category in expression (X.6) one would expect to write generally a membership $\mathbf{u} \in \varpi(p, q)$ instead of an equality $\mathbf{u} = \varpi(p, q)$. However, in the groupoid category of massive objects ϖ , the cardinality of $\varpi(p, q)$, for $\forall p, q \in \text{obj } \varpi$, by definition must be exactly one $|\varpi(p, q)| = 1$. There is the *unique* velocity of a massive body q relative to the reference system p . The category of observers ϖ actually is an enriched category, a kind of a (1, 1)-category. Every velocity-morphism $\varpi(p, q)$ must be an *object* in a category \mathbb{V} of all admissible relative velocities, $\text{obj } \mathbb{V} \equiv \{\varpi(p, q)\}$, and therefore ϖ actually is a functor

$$\varpi \times \varpi \xrightarrow{\varpi} \mathbb{V}. \quad (\text{X.10})$$

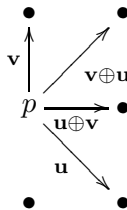
Each object of ϖ can be massive observer/source or observed/target: there is no division between observer and observed. However, in a velocity-morphism, $\varpi(p, q)$, a source object $p \in \text{obj } \varpi$, actually is an observer/laboratory, and a target object $q \in \text{obj } \varpi$, is a passive body being observed. Our philosophy is that the relative velocity is objective and independent of actual measurements.

In groupoid category of time-like vector fields (massive bodies) the *inverse* relative-velocity-morphism \mathbf{v}^{-1} is interior-observer-dependent, and not absolute as it is in the isometric-Lorentz-group formulation where, $\mathbf{v}^{-1} \equiv -\mathbf{v}$. This is because the relative velocity \mathbf{v} is g -orthogonal to an observer p , $\mathbf{v} \in \ker p$, $p\mathbf{v} = 0$. Somehow contrary to orthogonality of relative velocity, Cui in very interesting paper is emphasizing that rather the force \simeq acceleration, and not relative velocity, must be orthogonal to the particle-observer [Cui 2006], as it is in the electromagnetic theory with the accepted Lorentz force-form $f = i_P F$.

The Einstein isometric ternary relative velocities, \mathbf{v} and \mathbf{u} , see (VI.5), are \oplus -add-able with respect to the binary addition \oplus , *i.e.* $\exists \mathbf{v} \oplus \mathbf{u}$, if and only if these both velocities have the same preferred observing laboratory p as an implicit source as shown in Figure 1,

$$\text{source } \mathbf{u} = \text{source } \mathbf{v} = \text{source } (\mathbf{v} \oplus \mathbf{u}) = p. \quad (\text{X.11})$$

FIG. 1: The nonassociative addition \oplus of isometric relative velocities needs preferred laboratory p .



The Lorentz-boost-based non-associative addition \oplus of Einstein's isometric-velocities can be derived from the associative addition of velocities-morphism given by the groupoid category of observers ϖ .

It is believed that:

- The Lorentz isometry group of transformations of orthogonal/Lorentz (1+1+1+1)-frames of references is *exclusive* for non-observable absolute velocity.

- The Lobachevsky coset space of the Lorentz group $SO(1, 3)/SO(3)$ describe the space of relative velocities [Varicak 1910].
- Lorentz-group-invariant physical laws is the *highest* guideline of physics. The concepts of Lorentz-covariance and the Lorentz-invariance are discussed by Ivezic [2003] .
- Nonassociativity of the addition (VIII.4) of isometric relative velocities is necessary for physics, is unavoidable.

We believe instead that:

- The observer-independence \neq the Lorentz-group relativistic invariance. The observer-dependence is not the same as the Lorentz/Poincaré covariance. We think that Lorentz covariance and invariance will dwindle in importance.
- The change of the material reference system, (1+3)-split/observer, need not to be representation of the isometry.
- The isometric relative velocity (parameterizing Lorentz boost) needs a preferred observer (it is a ternary link). Maybe physics do not needs this?
- The binary relative velocities-morphisms are unique, and therefore are added associatively.

We are proposing the change of a concept of relative velocity, and this gives an alternative explanation for nonassociativity of the addition \oplus , and offer, among other, an alternative solution for the Mocanu paradox in terms of the unique relative velocities being the categorical morphisms.

The kinematics of the Lorentz-group-free relativity is ruled by an operator algebra generated by atomic split-idempotents, and denoted $\text{Obs}(\varpi)$. We believe that this operator algebra is a Frobenius algebra for each n -body problem, $\forall n \in \mathbb{N}$, however the full proof of this hypothesis is still missing.

A massive holonomic observer is a pair of the transverse equivalence relations in the space-time:

- Galileo Galilei (1564-1642) in his book in 1632 introduced the relation among events ‘to be in the same place’. This relation defines the one-dimensional congruence and

the codimension one (three-dimensional) physical space of locations = positions as the quotient space. To understand the relativity of the ‘location=place’ one must consider at least three massive events from the point of view of two massive observers: an observer on the beach, and a tourist on the traveling ship. One event alone possesses neither a ‘place’ nor a ‘moment’ of a time!

Following Brillouin [1970] we emphasize the fundamental importance of the non-zero-mass bodies for the formulation of the relativity of the space and for the relativity of the (proper)-time. This is in contradiction with wide spread opinion that the relativity theory must be primary related to basis laws of massless optics and of massless light propagation. That the Lorentz group is equivalent to the massless light propagation, e.g. [Bargmann 1957]. Such opinion originate not from Einstein formulation of special relativity in 1905, but from early Lorentz and Poincaré discovery in 1904 that the Lorentz group is a symmetry of the Maxwell equations. In fact the symmetry group of the Maxwell equations is *not* Lorentz group, and *not* Poincaré group, but the conformal group that does not found yet his proper place in the foundations of the relativity theory. We agree that posteriori relativity theory can and must be tested experimentally by means of the optical experiments, however we claim the uselessness of massless radiation for the definition of the Galilean relativity of space as the set of locations. The massless radiation can not be considered to be the reference system.

- Albert Einstein in 1905 introduced the simultaneity relation among events which define the relative proper time of the observer, and the 1-co-dimensional quotient-congruence, and one dimensional the Minkowski observer-dependent proper-time as the quotient [Minkowski 1908].

These two equivalence relations must be seen as the pair of transverse congruences, 1-dimensional and 1-co-dimensional: they need the massive body as the primary concept. Massless radiation does not possess the space-time split.

The Descartes and Newtonian motion, the relative velocity among massive bodies, presuppose the concepts of (empty and absolute) space (aether) and of time. If space is not absolute, then is the massive observer-dependent. No matter \implies no concept of the relative space. Therefore we wish to consider the concept of the matter and the relative motion as the primary concepts that do not need yet neither space nor time and no spacetime. Matter

first.

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