

# Second order equation of motion for electromagnetic radiation back-reaction

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## Abstract

We take the viewpoint that the physically acceptable solutions of the Lorentz–Dirac equation for radiation back-reaction are actually determined by a second order equation of motion in such a way that the self-force can be given as a function of spacetime location and velocity. This self-force function turns out to be determined by a first order partial differential equation. In view of possible practical difficulty in solving that partial differential equation, we propose two iteration methods, too, for obtaining the self-force function. For two example systems, the second order equation of motion is obtained exactly in the nonrelativistic regime via each of the three methods, and the three results are found to coincide. We reveal that, for both systems, back-reaction induces a damping proportional to velocity and, in addition, it decreases the effect of the external force.

**Keywords:** electromagnetic radiation, back-reaction, Lorentz–Dirac equation

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## 1 Introduction

The Maxwell equations of classical electrodynamics describe how *given* charged particles determine the electromagnetic field, while the Newton equation with the Lorentz force tells how a *given* electromagnetic field determines the motion of charged particles. As a first step towards describing *interaction* between matter and field, the radiation back-reaction (self-force) of a point charge  $e$  having the special relativistic world line function  $r$  is deduced to be ([2] Ch. 16.3, [3] Ch. III.3, [4], [5], [6], [7])

$$f_{\text{self}}^j = \eta (g^j_k - \dot{r}^j \dot{r}_k / c^2) \ddot{r}^k \quad (1)$$

where indices run from 0 to 3,  $g$  is the spacetime metric,  $c$  is the speed of light,  $\eta = (2/3)e^2/c^3$ , and overdot denotes differentiation with respect to proper time (note [1]). This force is added to the external force  $f$ , which may depend on both the spacetime location and the four-velocity of the particle, to obtain

$$m\ddot{r}^j = f^j(r, \dot{r}) + \eta (g^j_k - \dot{r}^j \dot{r}_k / c^2) \ddot{r}^k, \quad (2)$$

called the Lorentz–Dirac equation, for the motion of the point particle with mass  $m$ .

The problems with this equation are well-known. First, it is of third order so the initial values of spacetime position, velocity and acceleration are necessary to obtain the motion, and there is no apparent reasoning how to prescribe acceleration. Second, the equation admits ‘runaway’ solutions—motions accelerating exponentially in time—, and, third, it exhibits acausal behavior.

There are a number of attempts to treat these problems (see, e.g., [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18], research along the lines of [19], etc.), trying to find the physically acceptable solutions of the Lorentz–Dirac equation and giving further and further insight into the situation.

One of the proposed and most frequently applied approach is the following [20]. As a zeroth approximation, the equation without radiation is considered:

$$m\ddot{x}^j = f^j(x, \dot{x}). \quad (3)$$

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The third derivative is computed from this equation:

$$\ddot{x}^j = \frac{1}{m} \left( \frac{\partial f^j}{\partial x^k} \dot{x}^k + \frac{\partial f^j}{\partial \dot{x}^k} \ddot{x}^k \right). \quad (4)$$

Then the second derivative here is replaced by the rhs of (3) (divided by  $m$ ), an expression of lower order derivatives, leading to the following approximation of the self-force (1):

$$b_{\{1\}}^j(x, \dot{x}) := \eta \left( g^j_k - \frac{\dot{x}^j \dot{x}_k}{c^2} \right) \frac{1}{m} \left( \frac{\partial f^k}{\partial x^l} \dot{x}^l + \frac{1}{m} \frac{\partial f^k}{\partial \dot{x}^l} f^l(x, \dot{x}) \right). \quad (5)$$

This is added to the external force to derive an approximate second order equation of motion:

$$m\ddot{x}^j = f^j(x, \dot{x}) + b_{\{1\}}^j(x, \dot{x}). \quad (6)$$

In known examples,  $b_{\{1\}}^j(x, \dot{x})$  is found to be a damping dissipative term.

Another idea [11] is that the initial values of spacetime position, velocity and acceleration cannot be given independently, and one has to find a ‘critical manifold’ formed by those initial values which do not result in runaway solutions. In [11], it is stated that the critical manifold admits a second order differential equation for the physically acceptable motions but the actual form of such an equation is not given. Instead, by a singular perturbation, only a first approximation is provided, which results in (6).

Here, our object of interest is the exact form of the second order differential equation for the critical manifold. To explain how to arrive at a second order differential equation of motion, let us sharply distinguish the notion of equality from the notion of equation, both usually denoted by the same symbol  $=$ . *An equality is an assertion while an equation is a definition.* For instance, the equality  $\sin^2 + \cos^2 = 1$  asserts a relation between the functions  $\sin$  and  $\cos$ . On the other hand, the equation  $x^2 + 3x - 1 = 0$  defines the set of numbers  $x$  for which this relation holds.

An equation of motion must be an equation which *defines the set of all possible motions* under the given external circumstances. Formula (1) is an *equality*, making an assertion on the self-force, for a *given* motion. Accordingly, we accept that the world line function  $r$  of a charged particle in an exterior field satisfies *equality* (2). This, however, does not mean that (2) is the equation of motion of the particle: maybe many of its solutions are not possible motions. Indeed, the runaway solutions are not.

We consider the self-force as a function of spacetime position and velocity,

$$f_{\text{self}}^j = b^j(x, \dot{x}) \quad (7)$$

and so the exact second order differential *equation* of motion becomes

$$m\ddot{x}^j = f^j(x, \dot{x}) + b^j(x, \dot{x}). \quad (8)$$

The condition that the solutions of this equation must satisfy the Lorentz–Dirac *equality* yields a partial differential equation for determining the self-force function.

In what follows, we set up this partial differential equation. We propose two iterative methods, too, to obtain the self-force function and the corresponding second order equation of motion. Actually, the first step of one of these iterative methods corresponds to (6).

For demonstration, we investigate the partial differential equation and the iterative methods in quantitative detail, in the nonrelativistic approximation [21], treating two special cases. For both, we find that the three approaches give the same result.

## 2 The self-force function

For convenience, we introduce the shorthands

$$\hat{f} := f/m, \quad \hat{\eta} := \eta/m, \quad \hat{b} := b/m. \quad (9)$$

## 2.1 Differential equation for the self-force function

As said, we assume that the equation of motion of a radiating particle is of the form

$$\ddot{x}^j = \hat{f}^j(x, \dot{x}) + \hat{b}^j(x, \dot{x}), \quad (10)$$

where the second term on the rhs is the self-force as a function of spacetime position and four-velocity. In order to obtain its actual expression, we use a fixed-point-like property, as follows. Computing the third derivative from the expected equation of motion (10), substituting  $\ddot{x}$  in the obtained expression by the rhs of (10), and applying  $\hat{\eta}(g^j_k - \dot{x}^j \dot{x}_k/c^2)$  to the result, we have to recover the self-force:

$$\hat{b}^j = \hat{\eta} \left( g^j_k - \frac{\dot{x}^j \dot{x}_k}{c^2} \right) \left[ \frac{\partial(\hat{f}^k + \hat{b}^k)}{\partial x^l} \dot{x}^l + \frac{\partial(\hat{f}^k + \hat{b}^k)}{\partial \dot{x}^l} (\hat{f}^l + \hat{b}^l) \right]. \quad (11)$$

Naturally, one has to keep in mind that  $x$  and  $\dot{x}$  are to be understood as independent variables. We can display this fact in a more self-explaining way, writing (11) as

$$\hat{b}^j(x, u) = \hat{\eta} \left( g^j_k - \frac{u^j u_k}{c^2} \right) \left\{ \frac{\partial[\hat{f}^k + \hat{b}^k](x, u)}{\partial x^l} u^l + \frac{\partial[\hat{f}^k + \hat{b}^k](x, u)}{\partial u^l} [\hat{f}^l + \hat{b}^l](x, u) \right\}. \quad (12)$$

This is a first order partial differential equation (system) for the two-variable function(s)  $\hat{b}^j$ . Its solution is expected to contain an arbitrary free function; on physical grounds, one can impose some requirements, via which one can obtain the sought self-force function uniquely.

To formulate the fundamental condition, let us draw attention to that the self-force depends on the external force,  $b(x, u) = b_f(x, u)$ . It is evident that there is no self-force without external action; this fact can be taken into account in two ways.

First, we require that if  $f$  is zero in a neighborhood of a spacetime point  $x_0$  and four-velocity  $u_0$  then

$$b_f(x_0, u_0) = 0. \quad (13)$$

Second, it is plausible to expect that less action generates less reaction; consequently, we demand that if the external action tends to zero then the self-force must tend to zero, too. Specifically, we will consider the self-force function  $b_{\kappa f}$  for every  $0 \leq \kappa \leq 1$  and will impose, in the pointwise sense,

$$\lim_{\kappa \rightarrow 0} b_{\kappa f}(x, u) = 0. \quad (14)$$

A third natural assumption is that if the external field has a spacetime symmetry then the self-force function has the same symmetry. Namely, if  $f$  is invariant under a Poincaré transformation  $P$  (with the underlying Lorentz transformation  $L$ ), i.e.,  $L^{-1}f(Px, Lu) = f(x, u)$ , then the same invariance must hold for  $b_f$ , too.

## 2.2 Iteration of the radiation term

Equation (11) is, in general, a rather complicated system of partial differential equations so it is not easy to find its solutions. Hence, we look for other methods as well, to determine the self-force function.

An idea is suggested by (6) which, as said, cannot be an exact equation. We can consider it, however, as a first approximation. Then it is a straightforward idea that we take an analogous second approximation:  $\ddot{x}$  is computed as the derivative of

$$\ddot{x}^j = \hat{f}^j(x, \dot{x}) + \hat{b}_{\{1\}}^j(x, \dot{x}), \quad (15)$$

and then  $\ddot{x}$  is substituted by the rhs of (15). Thus, we obtain an expression  $\hat{b}_{\{2\}}^j(x, \dot{x})$ , with which the second approximation for the equation of motion is

$$\ddot{x}^j = \hat{f}^j(x, \dot{x}) + \hat{b}_{\{2\}}^j(x, \dot{x}). \quad (16)$$

The same procedure can be repeated iteratively for all higher orders. If the sequence of terms  $\hat{b}_{\{n\}}^j(x, \dot{x})$  converges to a  $\hat{b}(x, \dot{x})$ —in some appropriate sense, e.g., pointwise—then we arrive at a second order equation of motion of the form (10). Naturally, it is a tough problem is whether the sequence converges or not.

### 2.3 Iteration of the solution

The above iteration method suggests another one, which would not directly result in an equation of motion but in the motion corresponding to initial values  $x_0$  of spacetime position and  $u_0$  of four-velocity.

Let the solution of the zeroth approximation (3) corresponding to initial values  $x_0$  and  $u_0$  be denoted by  $r_{\{0\}}$ . Taking its first and third derivatives, we establish the differential equation

$$\ddot{x}^j = \hat{f}^j(x, \dot{x}) + \hat{\eta} \left( g^j_k - \dot{r}_{\{0\}}^j \dot{r}_{\{0\}k} / c^2 \right) \ddot{r}_{\{0\}}^k \quad (17)$$

as the first approximation for the equation of motion. Let  $r_{\{1\}}$  be its solution for initial values  $x_0$  and  $u_0$ . Taking its first and third derivative, we establish the second approximation, and so on; at step  $n + 1$  we solve

$$\ddot{x}^j = \hat{f}^j(x, \dot{x}) + \hat{\eta} \left( g^j_k - \dot{r}_{\{n\}}^j \dot{r}_{\{n\}k} / c^2 \right) \ddot{r}_{\{n\}}^k. \quad (18)$$

If the sequence of solutions  $r_{\{n\}}$  converges to an  $r$  then we succeeded in finding a motion satisfying equality (2) without the need for the initial value of acceleration. This motion is, therefore, a good candidate for the sought physical solution.

Naturally, here, too, convergence is a tough problem.

We can observe that, although this iterative method provides solutions rather than the equation of motion, a corresponding self-force function  $b(x, u)$  can be read off from the solutions. Namely, the value of the corresponding  $\hat{b}$  at any spacetime point  $x_0$  and four-velocity value  $u_0$  can be calculated from the third derivative of the solution  $r$  belonging to initial values  $x_0$  and  $u_0$ , at the initial proper time value:

$$\hat{b}^j(x_0, u_0) = \hat{\eta} \left( g^j_k - (u_0)^j (u_0)_k / c^2 \right) \ddot{r}^k(0). \quad (19)$$

## 3 Applying the three approaches in the nonrelativistic regime: Constant field

Let us consider a constant external electromagnetic field, which acts on the charged particle via the Lorentz force

$$\mathbf{f} = e\mathbf{E} + e\mathbf{v} \times \mathbf{B} = e\mathbf{E} + e\mathbf{F}\mathbf{v}, \quad (20)$$

where  $\mathbf{E}$  is the electric field three-vector,  $\mathbf{B}$  is the magnetic axial vector field,  $\mathbf{F} = (-\mathbf{B} \times)$  is the corresponding antisymmetric three-tensor—now each assumed space and time independent—, and  $\mathbf{v}$  is the velocity of the particle. With the shorthands

$$\tilde{\mathbf{E}} := \frac{e}{m}\mathbf{E}, \quad \tilde{\mathbf{B}} := \frac{e}{m}\mathbf{B}, \quad \tilde{\mathbf{F}} := \frac{e}{m}\mathbf{F}, \quad (21)$$

we can simply write

$$\hat{\mathbf{f}} = \hat{\mathbf{f}}(\mathbf{v}) = \tilde{\mathbf{E}} + \tilde{\mathbf{F}}\mathbf{v}. \quad (22)$$

The electromagnetic nature of the field will not play any role here so the subsequent considerations will be applicable for any force of the form (22), including a constant gravitational attraction, for example.

We will need some technical remarks regarding  $\tilde{\mathbf{F}}$ . Its kernel is spanned by  $\tilde{\mathbf{B}}$ , its range is the plane orthogonal to  $\tilde{\mathbf{B}}$ , and, with  $\mathbf{P}$  denoting the orthogonal projection onto this plane, we find

$$\tilde{\mathbf{F}} = \mathbf{P}\tilde{\mathbf{F}} = \tilde{\mathbf{F}}\mathbf{P} = \mathbf{P}\tilde{\mathbf{F}}\mathbf{P}, \quad (\mathbf{I} - \mathbf{P})\tilde{\mathbf{F}} = \mathbf{0}, \quad \tilde{\mathbf{F}}^2 = -\tilde{\mathbf{B}}^2\mathbf{P}, \quad (23)$$

where  $\mathbf{I}$  stands for the three-identity tensor (and  $\tilde{\mathbf{B}}$  is the magnitude of  $\tilde{\mathbf{B}}$ ).

To keep the formulae shorter and more easily accessible, we first treat  $\tilde{\mathbf{E}} = \mathbf{0}$ . In this case, the equation of motion without radiation is

$$\dot{\mathbf{v}} = \tilde{\mathbf{F}}\mathbf{v}. \quad (24)$$

The results for the general case  $\tilde{\mathbf{E}} \neq \mathbf{0}$  are presented in section 3.4.

### 3.1 Differential equation for the self-force function

We can start with ruling out the space and time dependence of  $\hat{\mathbf{b}}$ , based on the requirement of section 2.1 that a spacetime symmetry of the external force should be respected by the self-force, too. In the present case, the symmetry in question is spacetime translation invariance. Hence, the sought equation of motion is of the form

$$\dot{\mathbf{v}} = \tilde{\mathbf{F}}\mathbf{v} + \hat{\mathbf{b}}(\mathbf{v}). \quad (25)$$

In addition, the external field is invariant for space inversion,  $-\tilde{\mathbf{F}}(-\mathbf{v}) = \tilde{\mathbf{F}}(\mathbf{v})$ , and thus  $-\hat{\mathbf{b}}(-\mathbf{v}) = \hat{\mathbf{b}}(\mathbf{v})$  is required, too.

According to our assumption described in section 2.1,  $\hat{\mathbf{b}}$  must obey the differential equation

$$\hat{\mathbf{b}}(\mathbf{v}) = \hat{\eta}[\tilde{\mathbf{F}} + \hat{\mathbf{b}}'(\mathbf{v})] [\tilde{\mathbf{F}}\mathbf{v} + \hat{\mathbf{b}}(\mathbf{v})], \quad (26)$$

with  $'$  denoting the derivative map. Let us assume that we can expand  $\hat{\mathbf{b}}$  in a series. Because of the space inversion symmetry, the even powers are zero, so

$$\hat{\mathbf{b}}(\mathbf{v}) = \mathbf{L}_1\mathbf{v} + \mathbf{L}_3(\mathbf{v}, \mathbf{v}, \mathbf{v}) + \mathbf{L}_5(\mathbf{v}, \mathbf{v}, \mathbf{v}, \mathbf{v}, \mathbf{v}) + \dots \quad (27)$$

where  $\mathbf{L}_1$  is a linear map,  $\mathbf{L}_3$  is a symmetric trilinear map etc.; keep in mind that they depend on  $\tilde{\mathbf{F}}$ . Using the notation  $\mathbf{L}_3(\mathbf{v}^3) := \mathbf{L}_3(\mathbf{v}, \mathbf{v}, \mathbf{v})$  etc., we obtain

$$\begin{aligned} \mathbf{L}_1\mathbf{v} + \mathbf{L}_3(\mathbf{v}^3) + \mathbf{L}_5(\mathbf{v}^5) + \dots &= \hat{\eta}[\tilde{\mathbf{F}} + \mathbf{L}_1 + 3\mathbf{L}_3(\mathbf{v}^2, \cdot) + 5\mathbf{L}_5(\mathbf{v}^4, \cdot) + \dots] \\ &\quad \times [\tilde{\mathbf{F}}\mathbf{v} + \mathbf{L}_1\mathbf{v} + \mathbf{L}_3(\mathbf{v}^3) + \mathbf{L}_5(\mathbf{v}^5) + \dots], \end{aligned} \quad (28)$$

from which it follows, order by order, that

$$\mathbf{L}_1 = \hat{\eta}(\tilde{\mathbf{F}} + \mathbf{L}_1)^2, \quad (29)$$

$$\mathbf{L}_3(\mathbf{v}^3) = \hat{\eta}[3\mathbf{L}_3(\mathbf{v}^2, (\tilde{\mathbf{F}} + \mathbf{L}_1)\mathbf{v}) + (\tilde{\mathbf{F}} + \mathbf{L}_1)\mathbf{L}_3(\mathbf{v}^3)], \quad (30)$$

$$\mathbf{L}_5(\mathbf{v}^5) = \hat{\eta}[3\mathbf{L}_3(\mathbf{v}^2, \mathbf{L}_3(\mathbf{v}^2)) + 5\mathbf{L}_5(\mathbf{v}^4, (\tilde{\mathbf{F}} + \mathbf{L}_1)\mathbf{v}) + (\tilde{\mathbf{F}} + \mathbf{L}_1)\mathbf{L}_5(\mathbf{v}^5)], \quad (31)$$

etc. Multiplying (29) from the left and from the right by  $\tilde{\mathbf{F}} + \mathbf{L}_1$ , we find that  $\mathbf{L}_1\tilde{\mathbf{F}} = \tilde{\mathbf{F}}\mathbf{L}_1$ . Then it is a simple algebraic fact that

$$\mathbf{L}_1 = -\alpha\tilde{\mathbf{F}} - \gamma\mathbf{P} + \beta(\mathbf{I} - \mathbf{P}), \quad (32)$$

where  $\alpha$ ,  $\gamma$  and  $\beta$  are scalar coefficients depending on  $\tilde{\mathbf{F}}$ . Applying the second condition put in section 2.1,  $\gamma$  and  $\beta$  must tend to zero if  $\tilde{\mathbf{F}}$  tends to zero.

Having (32), (29) yields

$$-\alpha\tilde{\mathbf{F}} - \gamma\mathbf{P} + \beta(\mathbf{I} - \mathbf{P}) = \hat{\eta}[-(1 - \alpha)^2\tilde{\mathbf{B}}^2\mathbf{P} + \gamma^2\mathbf{P} + \beta^2(\mathbf{I} - \mathbf{P}) - 2\gamma(1 - \alpha)\tilde{\mathbf{F}}], \quad (33)$$

which tells

$$\alpha = 2\hat{\eta}\gamma(1 - \alpha), \quad \gamma = \hat{\eta}(1 - \alpha)^2\tilde{\mathbf{B}}^2 - \hat{\eta}\gamma^2 \quad (34)$$

as well as  $\beta = \hat{\eta}\beta^2$ , according to which  $\beta$  is either zero or equals  $1/\hat{\eta}$ . The second possibility is excluded by the demand that  $\beta$  must be zero for zero external field so

$$\beta = 0. \quad (35)$$

The first equation in (34) rules out  $\alpha = 1$ , and then another equivalent pair of equations is

$$\hat{\eta}\gamma = \frac{\alpha}{2(1 - \alpha)}, \quad 4(\hat{\eta}\tilde{\mathbf{B}})^2(1 - \alpha)^4 + (1 - \alpha)^2 - 1 = 0. \quad (36)$$

The latter condition is a quadratic equation for  $\varrho := (1 - \alpha)^2 \geq 0$ , with the only non-negative root

$$\varrho = \left(-1 + \sqrt{1 + 16(\hat{\eta}\tilde{\mathbf{B}})^2}\right) / \left(8(\hat{\eta}\tilde{\mathbf{B}})^2\right). \quad (37)$$

Again, the condition that  $\gamma$  must be zero for zero external field, gives the result:

$$\alpha = 1 - \sqrt{\varrho}, \quad \gamma = (1/\sqrt{\varrho} - 1) / (2\hat{\eta}), \quad (38)$$

the latter obtained from the first equation in (36). As it is proved in the Appendix, (30) yields  $\mathbf{L}_3 = \mathbf{0}$ , and, similarly, all the higher order terms are found to be zero. Hence, the self-force is

$$\hat{\mathbf{b}}(\mathbf{v}) = -(\alpha\tilde{\mathbf{F}} + \gamma\mathbf{P})\mathbf{v}, \quad (39)$$

and the equation of motion is

$$\dot{\mathbf{v}} = \tilde{\mathbf{F}}\mathbf{v} + \hat{\mathbf{b}}(\mathbf{v}), \quad \text{i.e.,} \quad \dot{\mathbf{v}} = [(1 - \alpha)\tilde{\mathbf{F}} - \gamma\mathbf{P}]\mathbf{v}. \quad (40)$$

It is informative to inspect the solution of this equation of motion, which is

$$\mathbf{w}(t) = (\mathbf{I} - \mathbf{P})\mathbf{v}_0 + e^{-\gamma t} e^{(1-\alpha)\tilde{\mathbf{F}}t} \mathbf{P}\mathbf{v}_0 \quad (41)$$

for initial velocity  $\mathbf{v}_0$  at zero initial time. For nonzero  $\tilde{B}$ , we have  $0 < 1 - \alpha < 1$  and  $\gamma > 0$  so radiation causes that

- the effect of the external magnetic field is reduced by a certain factor, and
- the component of velocity perpendicular to the magnetic field tends to zero as time passes.

Note that, in this simple case, two conditions given at the end of section 2.1 suffice to determine the self-force function completely.

### 3.2 Iteration of the radiation term

It follows from the equation without radiation—the zeroth approximation (24)—that  $\ddot{\mathbf{v}} = \tilde{\mathbf{F}}\dot{\mathbf{v}} = \tilde{\mathbf{F}}^2\mathbf{v} = -\tilde{B}^2\mathbf{P}\mathbf{v}$ . Accordingly, the first approximation of the radiation term is

$$\hat{\mathbf{b}}_{\{1\}}(\mathbf{v}) = \mathbf{L}_{\{1\}}\mathbf{v}, \quad \mathbf{L}_{\{1\}} = -\hat{\eta}\tilde{B}^2\mathbf{P}, \quad (42)$$

with which the first approximation of the equation of motion becomes  $\dot{\mathbf{v}} = (\tilde{\mathbf{F}} + \mathbf{L}_{\{1\}})\mathbf{v}$ . The radiation term derived from this equation equals

$$\hat{\mathbf{b}}_{\{2\}}(\mathbf{v}) = \mathbf{L}_{\{2\}}\mathbf{v}, \quad \mathbf{L}_{\{2\}} = \hat{\eta}(\tilde{\mathbf{F}} + \mathbf{L}_{\{1\}})^2. \quad (43)$$

Repeating this again and again, we recognize the generic recursion formula

$$\hat{\mathbf{b}}_{\{n\}}(\mathbf{v}) = \mathbf{L}_{\{n\}}\mathbf{v}, \quad \mathbf{L}_{\{n\}} = \hat{\eta}(\tilde{\mathbf{F}} + \mathbf{L}_{\{n-1\}})^2. \quad (44)$$

Since

$$\mathbf{L}_{\{2\}} = \hat{\eta}(\tilde{\mathbf{F}} + \mathbf{L}_{\{1\}})^2 = \hat{\eta}(-\tilde{B}^2\mathbf{P} - 2\hat{\eta}\tilde{B}^2\tilde{\mathbf{F}} + \hat{\eta}^2\tilde{B}^4\mathbf{P}), \quad (45)$$

every  $\mathbf{L}_{\{n\}}$  is a linear combination of  $\tilde{\mathbf{F}}$  and  $\mathbf{P}$ . Supposing that the sequence  $\hat{\mathbf{b}}_{\{n\}}(\mathbf{v})$  converges for all  $\mathbf{v}$ ,  $\mathbf{L}_{\{n\}}$  should converge to an  $\mathbf{L}$ , for which we have

$$\mathbf{L} = \hat{\eta}(\tilde{\mathbf{F}} + \mathbf{L})^2, \quad (46)$$

where  $\mathbf{L}$  is a linear combination of  $\tilde{\mathbf{F}}$  and  $\mathbf{P}$ :  $\mathbf{L} = -\alpha\tilde{\mathbf{F}} - \gamma\mathbf{P}$ . Consequently,

$$-\alpha\tilde{\mathbf{F}} - \gamma\mathbf{P} = \hat{\eta} \left[ -(1 - \alpha)^2\tilde{B}^2\mathbf{P} - 2\gamma(1 - \alpha)\tilde{\mathbf{F}} + \gamma^2\mathbf{P} \right], \quad (47)$$

which imposes the pair of equations in (34), i.e. we arrive at the same result as previously.

The ambiguity  $\alpha = 1 \pm \sqrt{\varrho}$  arises here, too. If convergence holds then, naturally, only one of the possibilities can be the limit. To select the correct one, we can consider the simple case when there is no external force. Then all the iteration terms are zero, and the iteration converges trivially. Hence, the coefficients must be zero for zero magnetic field, as previously.

A problem with the method of iterating the radiation term is that it is hard to obtain conditions for the convergence of the iteration. Unfortunately, convergence may not hold. Indeed, for  $\hat{\eta}\tilde{B} = 1$ —i.e., for such special magnetic fields—convergence does not occur, since then  $\mathbf{L}_{\{2\}} = -2\tilde{\mathbf{F}}$ , so  $\mathbf{L}_{\{3\}} = \mathbf{L}_{\{1\}}$ . Thus, for all higher  $n$ s,

$$\mathbf{L}_{\{n\}} = \begin{cases} \mathbf{L}_{\{2\}} & (n \text{ is even}), \\ \mathbf{L}_{\{1\}} & (n \text{ is odd}). \end{cases} \quad (48)$$

It is interesting, however, that the final result (38) does not exclude the case  $\hat{\eta}\tilde{B} = 1$ , and provides solution for  $\hat{\eta}\tilde{B} > 1$  as well (which is expected to be outside the domain of convergence).

### 3.3 Iteration of the solution

Considering an initial value  $\mathbf{v}_0$  at zero initial time, the zeroth equation (24) has the solution

$$\mathbf{w}_{\{0\}}(t) = e^{\tilde{\mathbf{F}}t} \mathbf{v}_0. \quad (49)$$

Then  $\ddot{\mathbf{w}}_{\{0\}}(t) = -\tilde{B}^2 e^{\tilde{\mathbf{F}}t} \mathbf{v}_0$ , so the first approximation satisfies the equation

$$\dot{\mathbf{v}} = \tilde{\mathbf{F}}\mathbf{v} - \hat{\eta}\tilde{B}^2 e^{\tilde{\mathbf{F}}t} \mathbf{v}_0, \quad (50)$$

whose solution with initial value  $\mathbf{v}_0$  is

$$\mathbf{w}_{\{1\}}(t) = \left(1 - \hat{\eta}\tilde{B}^2 t\right) e^{\tilde{\mathbf{F}}t} \mathbf{v}_0. \quad (51)$$

Then  $\ddot{\mathbf{w}}_{\{1\}}(t) = -\left[\tilde{B}^2 \left(1 - \hat{\eta}\tilde{B}^2 t\right) + 2\hat{\eta}\tilde{B}^2 \tilde{\mathbf{F}}\right] e^{\tilde{\mathbf{F}}t} \mathbf{v}_0$ . The second approximation satisfies the equation

$$\dot{\mathbf{v}} = \tilde{\mathbf{F}}\mathbf{v} - \hat{\eta} \left[ \tilde{B}^2 \left(1 - \hat{\eta}\tilde{B}^2 t\right) + 2\hat{\eta}\tilde{B}^2 \tilde{\mathbf{F}} \right] e^{\tilde{\mathbf{F}}t} \mathbf{v}_0, \quad (52)$$

with the solution

$$\mathbf{w}_{\{2\}}(t) = \left[ 1 - \hat{\eta}\tilde{B}^2 \left( t - \frac{1}{2}\hat{\eta}\tilde{B}^2 t \right) - 2\hat{\eta}^2 \tilde{B}^2 t \tilde{\mathbf{F}} \right] e^{\tilde{\mathbf{F}}t} \mathbf{v}_0. \quad (53)$$

At the  $n$ th step of this iteration scheme, we find

$$\mathbf{w}_{\{n\}}(t) = \left[ p_{\{n\}}(t) + q_{\{n\}}(t) \tilde{\mathbf{F}} \right] e^{\tilde{\mathbf{F}}t} \mathbf{v}_0, \quad (54)$$

where  $p_{\{n\}}$  and  $q_{\{n\}}$  are polynomials of  $t$  satisfying the recursive formulae

$$\dot{p}_{\{n\}} = \hat{\eta} \left( \ddot{p}_{\{n-1\}} - 2\tilde{B}^2 \dot{q}_{\{n-1\}} - \tilde{B}^2 p_{n-1} \right), \quad (55)$$

$$\dot{q}_{\{n\}} = \hat{\eta} \left( \ddot{q}_{\{n-1\}} + 2\dot{p}_{\{n-1\}} - \tilde{B}^2 q_{\{n-1\}} \right) \quad (56)$$

and the initial conditions  $p_{\{n\}}(0) = 1$ ,  $q_{\{n\}}(0) = 0$ .

Let us suppose that  $p := \lim_n p_{\{n\}}$  and  $q := \lim_n q_{\{n\}}$  exist and, moreover, that the limit procedure and differentiation can be interchanged. Then, for  $\mathbf{w} := \lim_n \mathbf{w}_{\{n\}}$ ,

$$\mathbf{w}(t) = [p(t) + q(t) \tilde{\mathbf{F}}] e^{\tilde{\mathbf{F}}t} \mathbf{v}_0, \quad (57)$$

$$\dot{p} = \hat{\eta} \left( \ddot{p} - 2\tilde{B}^2 \dot{q} - \tilde{B}^2 p \right), \quad (58)$$

$$\dot{q} = \hat{\eta} \left( \ddot{q} + 2\dot{p} - \tilde{B}^2 q \right) \quad (59)$$

together with the conditions  $p(0) = 1$ ,  $q(0) = 0$ . With the notation  $s := p + i\tilde{B}q$ , (58) and (59) can be comprised as

$$\ddot{s} - \left( 1/\hat{\eta} - 2i\tilde{B} \right) \dot{s} - \tilde{B}^2 s = 0. \quad (60)$$

The roots of the corresponding characteristic polynomial are

$$1/2\hat{\eta} - i\tilde{B} \pm \sqrt{1/(4\hat{\eta}^2) - i\tilde{B}/\hat{\eta}}. \quad (61)$$

The emerging ambiguity can be resolved as previously: if the magnetic force is sent to zero then  $p_{\{n\}}(t) = 1$  for all  $n$ , therefore, we choose the root that vanishes for  $\tilde{B} \rightarrow 0$  so as to obtain the solution  $p(t) = 1$ , i.e., the free motion (no self-force).

Therefore, if  $(-\gamma)$  and  $\alpha$  are the real and the imaginary part of the root, respectively, then the solution of our system of differential equations is

$$p(t) = e^{-\gamma t} \cos(\alpha \tilde{B} t), \quad q(t) = -e^{-\gamma t} \sin(\alpha \tilde{B} t) / \tilde{B}, \quad (62)$$

where  $\alpha$  and  $\gamma$  satisfy the relations in (34).

We have obtained the same result as previously. In particular, utilizing (23), one can demonstrate that (57) is the same as (41). At last, it is straightforward to find that the self-force function corresponding to (57)—see (19)—proves to be the same as (39).

On the other side, one can also observe that, though leading to the same result, the two iterations themselves are different: the solutions of the iterated equations  $\dot{\mathbf{v}} = (\tilde{\mathbf{F}} + \mathbf{L}_{\{n\}})\mathbf{v}$  do not equal the functions (54).

Convergence is a nontrivial question in this approach, too. Nevertheless, it is interesting that, with this method, there is no evidence for excluding the case  $\hat{\eta}\tilde{B} = 1$ .

### 3.4 Nonzero electric and magnetic field

As anticipated in section 3, now we turn towards the case of a nonzero constant electric field in addition to the nonzero constant magnetic field.

It is beneficial to decompose the electric field into components parallel to and orthogonal to the magnetic field, respectively, and to observe that this decomposition can be written in the form

$$\tilde{\mathbf{E}} = (\mathbf{I} - \mathbf{P})\tilde{\mathbf{E}} + \tilde{\mathbf{F}} \left( -\frac{1}{\tilde{B}^2} \tilde{\mathbf{F}}\tilde{\mathbf{E}} \right). \quad (63)$$

This induces a decomposition of the equation of motion without self-force [i.e., containing only the external Lorentz force (22)]. The  $\tilde{\mathbf{B}}$ -parallel component,

$$[(\mathbf{I} - \mathbf{P})\mathbf{v}]' = (\mathbf{I} - \mathbf{P})\tilde{\mathbf{E}}, \quad (64)$$

governs only the  $\tilde{\mathbf{B}}$ -parallel component of  $\mathbf{v}$ , while the  $\tilde{\mathbf{B}}$ -orthogonal part can be written as

$$\dot{\mathbf{u}} = \tilde{\mathbf{F}}\mathbf{u} \quad \text{with} \quad \mathbf{u} := \mathbf{P}\mathbf{v} - \frac{1}{\tilde{B}^2} \tilde{\mathbf{F}}\tilde{\mathbf{E}}, \quad \mathbf{u} \perp \tilde{\mathbf{B}}, \quad (65)$$

and determines the time evolution of the  $\tilde{\mathbf{B}}$ -orthogonal component of  $\mathbf{v}$ .

Now we add the self-force term. Both iteration methods, namely, that of the radiation term and that of the solution, can be found to preserve this decomposition, where the  $\tilde{\mathbf{B}}$ -orthogonal part can actually be treated the same way for  $\mathbf{u}$  as we proceeded in the  $\tilde{\mathbf{E}} = \mathbf{0}$  case for  $\mathbf{v}$ . Both approaches provide

$$\hat{\mathbf{b}}(\mathbf{v}) = -\alpha\mathbf{P}\tilde{\mathbf{E}} + \gamma\frac{1}{\tilde{B}^2}\tilde{\mathbf{F}}\tilde{\mathbf{E}} - [\alpha\tilde{\mathbf{F}} + \gamma\mathbf{P}]\mathbf{v} \quad (66)$$

for the self-force and

$$\mathbf{v}(t) = (\mathbf{I} - \mathbf{P})\tilde{\mathbf{E}}t + (\mathbf{I} - \mathbf{P})\mathbf{v}_0 + \frac{1}{\tilde{B}^2}\tilde{\mathbf{F}}\tilde{\mathbf{E}} + e^{-\gamma t}e^{(1-\alpha)\tilde{\mathbf{F}}t} \left( \mathbf{P}\mathbf{v}_0 - \frac{1}{\tilde{B}^2}\tilde{\mathbf{F}}\tilde{\mathbf{E}} \right) \quad (67)$$

for the solution, after putting the two decomposed parts together. The found self-force function proves to be a solution of the partial differential equation of the third approach as well, and satisfies the two additional requirements—vanishing for vanishing external field and symmetry preservation.

## 4 Applying the three approaches in the nonrelativistic regime: Harmonic force

As our second physical system considered as example for the three proposed methods, we next investigate the one-dimensional nonrelativistic motion due to a harmonic elastic force. Without radiation back-reaction, the equation is

$$\ddot{x} = -\omega^2 x, \quad (68)$$

where  $\omega$  is a non-negative constant.



## 4.1 Differential equation for the self-force function

According to our assumption described in section 2.1, the radiation self-force function  $\hat{b}(x, \dot{x})$  in the anticipated equation of motion

$$\ddot{x} = -\omega^2 x + \hat{b}(x, \dot{x}) \quad (69)$$

is to satisfy the quasi-linear partial differential equation

$$\hat{b}(x, v) = \hat{\eta} \left[ -\omega^2 v + \frac{\partial \hat{b}(x, v)}{\partial x} v + \frac{\partial \hat{b}(x, v)}{\partial v} (-\omega^2 x + \hat{b}) \right]. \quad (70)$$

Note that, although not denoted explicitly, the sought  $\hat{b}$  also depends on  $\omega$ , i.e., on the external force.

The characteristic ordinary differential equation corresponding to (70) reads

$$\frac{dx}{d\xi} = v, \quad \frac{dv}{d\xi} = -\omega^2 x + \hat{b}, \quad \frac{d\hat{b}}{d\xi} = \frac{1}{\hat{\eta}} \hat{b} + \omega^2 v. \quad (71)$$

This is a simple linear differential equation whose characteristic roots  $\lambda$  fulfill the equation

$$\hat{\eta} \lambda^3 - \lambda^2 - \omega^2 = 0. \quad (72)$$

We can find its solutions (roots) by the Cardano formula. With the notation

$$\rho_{\pm} := \sqrt[3]{\hat{\eta}^2 \omega^2 / 2 + 1/27 \pm \sqrt{\hat{\eta}^4 \omega^4 / 4 + \hat{\eta}^2 \omega^2 / 27}}, \quad (73)$$

the roots are

$$\hat{\eta} \lambda_1 = \frac{1}{3} - \frac{\rho_+ + \rho_-}{2} + i \frac{\sqrt{3}(\rho_+ - \rho_-)}{2}, \quad (74)$$

$$\hat{\eta} \lambda_2 = \frac{1}{3} - \frac{\rho_+ + \rho_-}{2} - i \frac{\sqrt{3}(\rho_+ - \rho_-)}{2}, \quad (75)$$

$$\hat{\eta} \lambda_3 = \frac{1}{3} + \rho_+ + \rho_-. \quad (76)$$

With the three roots, the solutions of equation (71) are of the form

$$x(\xi) = \sum_{i=1}^3 c_i e^{\lambda_i \xi}, \quad v(\xi) = \sum_{i=1}^3 \lambda_i c_i e^{\lambda_i \xi}, \quad \hat{b}(\xi) = \sum_{i=1}^3 \lambda_i^2 c_i e^{\lambda_i \xi} + \omega^2 x(\xi). \quad (77)$$

According to the method of characteristics, via eliminating the auxiliary variable  $\xi$ , one obtains  $\hat{b}$  as a function of  $x$  and  $v$ . Now we use the condition of section 2.1 that  $\hat{b}$  must be zero for zero external force i.e. for  $\omega = 0$ . Since  $\lambda_1$  and  $\lambda_2$  are zero for  $\omega = 0$  and  $\lambda_3$  is not zero, we deduce from the last equality above that  $c_3 = 0$  is necessary.

Then  $\xi$  can be eliminated easily; with the notations  $z_1 := c_1 e^{\lambda_1 \xi}$  and  $z_2 := c_2 e^{\lambda_2 \xi}$ , we have

$$x = z_1 + z_2, \quad v = \lambda_1 z_1 + \lambda_2 z_2, \quad \hat{b} = \lambda_1^2 z_1 + \lambda_2^2 z_2 + \omega^2 x. \quad (78)$$

Here, the first two equations enable one to express  $z_1$  and  $z_2$  as a linear function of  $x$  and  $v$ . Substituting them into the third equation provides  $\hat{b}$  also as a linear function of  $x$  and  $v$ :  $\hat{b}(x, v) = (\omega^2 - \lambda_1 \lambda_2) x + (\lambda_1 + \lambda_2) v$ .

A convenient way to proceed is to write the coefficients in another form:

$$\hat{b}(x, v) = \alpha \omega^2 x - \gamma v. \quad (79)$$

Evidently,

$$\hat{\eta} \gamma = -\frac{2}{3} + \rho_+ + \rho_-, \quad (80)$$

and, substituting (79) into (70), we obtain

$$\alpha = \frac{\hat{\eta} \gamma}{1 + \hat{\eta} \gamma} \quad \text{and} \quad \hat{\eta} \gamma (1 + \hat{\eta} \gamma)^2 = \hat{\eta}^2 \omega^2. \quad (81)$$

We can see that  $\gamma > 0$  and  $0 < \alpha < 1$  are necessary for  $\omega \neq 0$ . Hence, the equation of motion with the radiation term (79) reads

$$\ddot{x} = -(1 - \alpha) \omega^2 x - \gamma \dot{x}. \quad (82)$$

According to this equation, radiation causes that

- the effect of the harmonic force is reduced by a certain factor, and
- the motion is damped by a term proportional to velocity.

We can see in this case, too, that the condition that the self-force must be zero if the external force is zero suffices to determine the self-force function completely.

## 4.2 Iteration of the radiation term

It follows from the equation without radiation (the zeroth approximation) (68) that  $\ddot{x} = -\omega^2 x$ , so the first approximation of the radiation term is

$$\hat{b}_{\{1\}}(x, \dot{x}) = -\hat{\eta}\omega^2 \dot{x}, \quad (83)$$

and the first approximate equation of motion becomes  $\ddot{x} = -\omega^2 x + \hat{b}_{\{1\}}(x, \dot{x})$ . Computing  $\ddot{x}$  from this equation and then replacing  $\ddot{x}$  with  $-\omega^2 x + \hat{b}_{\{1\}}(x, \dot{x})$ , we obtain the second approximation

$$\hat{b}_{\{2\}}(x, \dot{x}) = \hat{\eta}^2 \omega^4 x - \hat{\eta}\omega^2 (1 - \hat{\eta}^2 \omega^2) \dot{x} \quad (84)$$

and the second equation  $\ddot{x} = -\omega^2 x + \hat{b}_{\{2\}}(x, \dot{x})$ .

It is straightforward then that the  $n$ th approximation is of the form

$$\hat{b}_{\{n\}}(x, \dot{x}) = \alpha_n \omega^2 x - \gamma_n \dot{x}, \quad (85)$$

where  $\alpha_n$  and  $\hat{\eta}\gamma_n$  are functions (polynomials) of  $\hat{\eta}^2 \omega^2$  and satisfy the recursive formulae

$$\alpha_{n+1} = \hat{\eta}\gamma_n (1 - \alpha_n), \quad \hat{\eta}\gamma_{n+1} = \hat{\eta}^2 \omega^2 (1 - \alpha_n) - (\hat{\eta}\gamma_n)^2. \quad (86)$$

Supposing that the sequence  $\hat{b}_{\{n\}}$  converges to a  $\hat{b}$ , i.e., the limits  $\alpha := \lim_n \alpha_n$  and  $\gamma := \lim_n \gamma_n$  exist, then

$$\hat{b}(x, \dot{x}) = \alpha \omega^2 x - \gamma \dot{x} \quad (87)$$

and

$$\alpha = \frac{\hat{\eta}\gamma}{1 + \hat{\eta}\gamma}, \quad \hat{\eta}\gamma(1 + \hat{\eta}\gamma)^2 = \hat{\eta}^2 \omega^2, \quad (88)$$

which coincide with (81). Putting  $\hat{\eta}\lambda := 1 + \hat{\eta}\gamma$ , the second equation above is transformed into the equation (72). Naturally, now we are interested in the real roots. We find that only one root is real, namely, with the notation (73),

$$\hat{\eta}\gamma = -\frac{2}{3} + \rho_+ + \rho_-. \quad (89)$$

Hence, we arrive at the same result as previously.

It is a problem, though, that it is difficult to obtain conditions for the convergence of the iteration. Unfortunately, convergence does not hold necessarily. Indeed, if  $\hat{\eta}\omega = 1$  then the sequence does not converge because then  $\hat{b}_{\{2\}}(x, \dot{x}) = \omega x^2$ , and thus  $\hat{b}_{\{3\}} = \hat{b}_{\{1\}}$ . Consequently,

$$\hat{b}_{\{n\}} = \begin{cases} \hat{b}_{\{2\}} & (n \text{ is even}), \\ \hat{b}_{\{1\}} & (n \text{ is odd}). \end{cases} \quad (90)$$

It is interesting, however, that, with (89), (88) provides a solution for all values of  $\hat{\eta}\omega$ .

## 4.3 Iteration of the solution

Considering some initial values  $x_0$  and  $v_0$  for position and velocity, the zeroth equation (68) has the solution

$$r_{\{0\}}(t) = z_0 e^{i\omega t} + z_0^* e^{-i\omega t} = z_0 e^{i\omega t} + \text{c. c.} \quad (91)$$

with

$$z_0 := (x_0 - i v_0 / \omega) / 2. \quad (92)$$

Since  $\ddot{r}_{\{0\}}(t) = -i z_0 \omega^3 e^{i\omega t} + \text{c. c.}$ , the equation for the first iterated solution reads

$$\ddot{x} = -\omega^2 x + \hat{\eta} (-i z_0 \omega^3 e^{i\omega t} + \text{c. c.}), \quad (93)$$

whose solution with the chosen initial values is

$$r_{\{1\}}(t) = [-(1/2)\hat{\eta}\omega^3 t + z_0] e^{i\omega t} + \text{c. c.} \quad (94)$$

Then we find that the  $n$ th solution is of the form

$$r_{\{n\}}(t) = p_n(t)e^{i\omega t} + \text{c. c.}, \quad (95)$$

where  $p_n(t)$  is a polynomial of  $t$  of  $n$ th degree; and we have

$$\ddot{r}_{\{n\}} = (\ddot{p}_n + 2i\omega\dot{p}_n - \omega^2 p_n)e^{i\omega t} + \text{c. c.} \quad (96)$$

and

$$\ddot{r}_{\{n\}} = (\ddot{p}_n + 3i\omega\dot{p}_n - 3\omega^2 p_n - i\omega^3 p_n)e^{i\omega t} + \text{c. c.}, \quad (97)$$

and the resulting recursive formula (arising both from the coefficient of  $e^{i\omega t}$  and from that of  $e^{-i\omega t}$ ) is

$$\ddot{p}_{n+1} + 2i\omega\dot{p}_{n+1} - \omega^2 p_{n+1} = -\omega^2 p_{n+1} + \hat{\eta}(\ddot{p}_n + 3i\omega\dot{p}_n - 3\omega^2 p_n - i\omega^3 p_n). \quad (98)$$

Thus, supposing convergence (and that differentiation can be interchanged with taking the limit), we find for the limit  $p := \lim_n p_n$

$$\ddot{p} + 2i\omega\dot{p} - \omega^2 p = -\omega^2 p + \hat{\eta}(\ddot{p} + 3i\omega\dot{p} - 3\omega^2 p - i\omega^3 p). \quad (99)$$

This is a linear differential equation of third order. Its solutions are of the form  $e^{\lambda t}$  with

$$(\lambda + i\omega)^2 + \omega^2 = \hat{\eta}(\lambda + i\omega)^3. \quad (100)$$

Putting  $\lambda + i\omega = \mu + i\nu$ , where  $\mu$  and  $\nu$  are real, we can rewrite this as

$$\mu^2 + 2i\mu\nu - \nu^2 + \omega^2 = \hat{\eta}(\mu^3 + 3i\mu^2\nu - 3\mu\nu^2 - i\nu^3), \quad (101)$$

which is equivalent to the pair of real equations

$$\nu^2 = 3\mu^2 - 2\mu/\hat{\eta}, \quad (102)$$

$$8(\hat{\eta}\mu)^3 - 8(\hat{\eta}\mu)^2 + 2(\hat{\eta}\mu) - (\hat{\eta}\omega)^2 = 0. \quad (103)$$

With  $\gamma := 2\mu$ , (103) reduces to the second equation of (88). In parallel, putting  $\nu^2 =: (1 - \alpha)\omega^2 - \gamma^2/4$  (i.e., defining  $\alpha$  in this way), we find  $\hat{\eta}\gamma(\hat{\eta}\gamma + 1) = (1 - \alpha)\omega^2$ , which, together with the second equation of (88), results in its first one.

As a consequence, whenever convergence holds, the iteration of solutions gives the same solutions and same self-force function as the iteration of the radiation term.

## 5 Discussion

We looked for a second order equation of motion whose solutions satisfy the Lorentz–Dirac equality and, at the same time, are physically acceptable. Our aim was to give back-reaction as a function of spacetime position and velocity. A simple argument showed that this self-force function is determined by a first order partial differential equation. Two iterative methods, too, were proposed for finding the self-force function.

In the nonrelativistic approximation we could exactly calculate the self-force function for two systems: a constant external electromagnetic field and a one-dimensional elastic external force. The three suggested methods turned out to lead to the same result. As concerns the physical picture, for both systems, radiation back-reaction has two manifestations: inducing a damping linear in velocity and reducing the strength of the external force.

The latter effect could also allow the—quantum field theory motivated—interpretation that back-reaction causes a positive renormalization of the mass of the particle. However, for the constant external electromagnetic field, this renormalization turns out to differ from a simple scalar multiplying of the mass. Rather, renormalization is a direction dependent, tensorial multiplication.

This system also proves to show a limitation of the criterion by Dirac and Haag [8, 9], which would choose that solution for initial position and velocity for which acceleration tends to zero for asymptotically large

times. In fact, this system is found to decouple into two independent subsystems, one parallel to and the other orthogonal to the magnetic field. In the former subsystem, acceleration remains time independent and corresponds to unrenormalized mass, while damping and renormalized mass (or renormalized external force) emerges in the latter subsystem.

It is therefore an important open question on what grounds the decrease of the external force can be interpreted as mass renormalization, and whether in other systems this renormalization is not only direction dependent but, for example, also (spacetime) position and velocity dependent (as suggested by some preliminary considerations not detailed here).

The three methods we proposed and investigated led to the same result for the two systems considered. Some differences among the three approaches were found, though. First, the problematic aspect is that there is an encoded amount of ambiguity in the partial differential equation to solve. This ambiguity was easy to rule out for the two systems we considered but may be a harder task for other systems. Second, in the two iteration approaches, convergence remained a tough open mathematical problem; moreover, not all coefficients of the model ensured the existence of a solution in one of the iteration methods.

Further study is needed, accordingly, about each method separately and also about some possible connections among them.

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## A The proof utilized in section 3.1

The claim that (30) yields  $\mathbf{L}_3 = \mathbf{0}$  can be proved as follows.

We start with deriving a simple algebraic fact. Let  $\mathbf{T}$  be a symmetric trilinear map. Then, with the simplifying notation used earlier,

$$\mathbf{T}((\mathbf{w} + \mathbf{v})^3) = \mathbf{T}(\mathbf{w}^3) + 3\mathbf{T}(\mathbf{w}^2, \mathbf{v}) + 3\mathbf{T}(\mathbf{w}, \mathbf{v}^2) + \mathbf{T}(\mathbf{v}^3) \quad (104)$$

for all  $\mathbf{v}$  and  $\mathbf{w}$ . Thus, if  $\mathbf{T}((\cdot)^3) = \mathbf{0}$  then  $3\mathbf{T}(\mathbf{w}^2, \mathbf{v}) + 3\mathbf{T}(\mathbf{w}, \mathbf{v}^2) = \mathbf{0}$  for all  $\mathbf{v}$  and  $\mathbf{w}$ . Here, for a fixed  $\mathbf{v}$ , the first term is bilinear, the second term is linear in  $\mathbf{w}$ ; their sum can be zero only if both are zero. As a consequence, if  $\mathbf{T}(\mathbf{v}^3) = \mathbf{0}$  for all  $\mathbf{v}$  then  $\mathbf{T}(\mathbf{w}^2, \mathbf{v}) = \mathbf{0}$  for all  $\mathbf{w}$  and  $\mathbf{v}$ . Further, for a fixed  $\mathbf{v}$ , we have

$$\mathbf{T}((\mathbf{w}_1 + \mathbf{w}_2)^2, \mathbf{v}) = \mathbf{T}(\mathbf{w}_1^2, \mathbf{v}) + 2\mathbf{T}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}) + \mathbf{T}(\mathbf{w}_2^2, \mathbf{v}), \quad (105)$$

which shows that, if  $\mathbf{T}(\mathbf{w}^2, \mathbf{v}) = \mathbf{0}$  for all  $\mathbf{w}$  and  $\mathbf{v}$ , then

$$\mathbf{T}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}) = \mathbf{0} \quad \text{for all } \mathbf{w}_1, \mathbf{w}_2, \mathbf{v}. \quad (106)$$

Hence, we have the result: if  $\mathbf{T}(\mathbf{v}^3) = \mathbf{0}$  for all  $\mathbf{v}$  then (106) holds.

The previous consideration and (30) yield that

$$[\mathbf{I} - \hat{\eta}(\tilde{\mathbf{F}} + \mathbf{L}_1)]\mathbf{L}_3(\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}) = 3\hat{\eta}\mathbf{L}_3(\mathbf{w}_1, \mathbf{w}_2, (\tilde{\mathbf{F}} + \mathbf{L}_1)\mathbf{v}) \quad (107)$$

for all  $\mathbf{w}_1, \mathbf{w}_2$  and  $\mathbf{v}$ . For fixed  $\mathbf{w}_1$  and  $\mathbf{w}_2$ ,  $\mathbf{W} := \mathbf{L}_3(\mathbf{w}_1, \mathbf{w}_2, \cdot)$  is a linear map satisfying

$$[\mathbf{I} - \hat{\eta}(\tilde{\mathbf{F}} + \mathbf{L}_1)]\mathbf{W} = 3\hat{\eta}\mathbf{W}(\tilde{\mathbf{F}} + \mathbf{L}_1), \quad (108)$$

which implies

$$\mathbf{W}[\mathbf{I} - 3\hat{\eta}(\tilde{\mathbf{F}} + \mathbf{L}_1)] = (\tilde{\mathbf{F}} + \mathbf{L}_1)\hat{\eta}\mathbf{W}, \quad (109)$$

too. Recall that  $\tilde{\mathbf{F}} + \mathbf{L}_1 = (1 - \alpha)\tilde{\mathbf{F}} - \gamma\mathbf{P}$ . Then it is a simple fact that the linear maps multiplying  $\mathbf{W}$  on the left hand side in (108) and (109), respectively, are nondegenerate. Therefore, applying  $\mathbf{I} - \mathbf{P}$  from the right to (108) and from the left to (109), we find  $\mathbf{W}\mathbf{P} = \mathbf{W} = \mathbf{P}\mathbf{W}$ .

As a consequence,  $\mathbf{W}$  commutes with  $\tilde{\mathbf{F}}$ , too, so it must be of the form  $\mathbf{W} = \beta\tilde{\mathbf{F}} + \zeta\mathbf{P}$ , and either (108) or (109) implies

$$(1 + 4\hat{\eta}\gamma)\mathbf{W} = 4\hat{\eta}(1 - \alpha)\mathbf{W}\tilde{\mathbf{F}}. \quad (110)$$

This equation and  $\mathbf{W} = \beta\tilde{\mathbf{F}} + \zeta\mathbf{P}$  result in

$$(1 + 4\hat{\eta}\gamma)\beta = 4\hat{\eta}(1 - \alpha)\zeta, \quad (1 + 4\hat{\eta}\gamma)\zeta = -4\hat{\eta}(1 - \alpha)\tilde{\mathbf{B}}^2\beta, \quad (111)$$

implying  $\tilde{\mathbf{B}}^2\beta^2 = -\zeta^2$ , which is possible only if  $\beta = 0$  and  $\zeta = 0$ , i.e.,  $\mathbf{W} = \mathbf{0}$ .

Since we had  $\mathbf{W} = \mathbf{L}_3(\mathbf{w}_1, \mathbf{w}_2, \cdot)$  for arbitrary  $\mathbf{w}_1$  and  $\mathbf{w}_2$ , we arrive at  $\mathbf{L}_3 = \mathbf{0}$ .

## References

- [1] Note that, upon  $\dot{r}^j \dot{r}_j = 1$ , we also have  $-\dot{r}^j \dot{r}_k \ddot{r}^k = \dot{r}^j \left[ \ddot{r}_k \ddot{r}^k - (1/2)(\dot{r}^k \dot{r}_k)'' \right] = \dot{r}^j \ddot{r}_k \ddot{r}^k$ . Frequently, the self-force is written using this latter form.
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- [21] Namely, we take only  $j = 1, 2, 3$  in (2) and omit terms of the order of  $1/c^2$  or higher.